MŰEGYETEM 1782

Tartószerkezetek Mechanikája Tanszék

Tartók statikája II

Tárcsák számítása

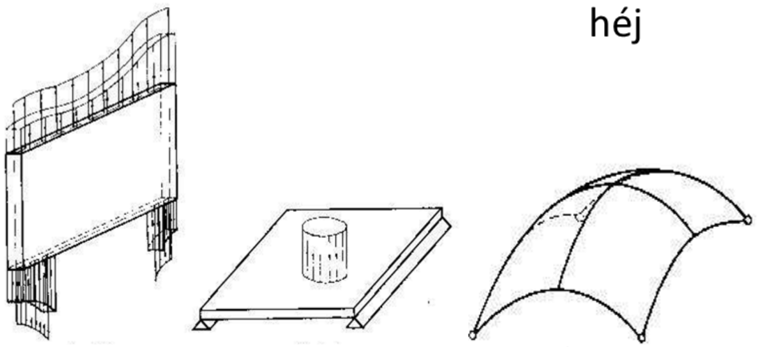
Dr. Hortobágyi Zsolt

Felületszerkezetek

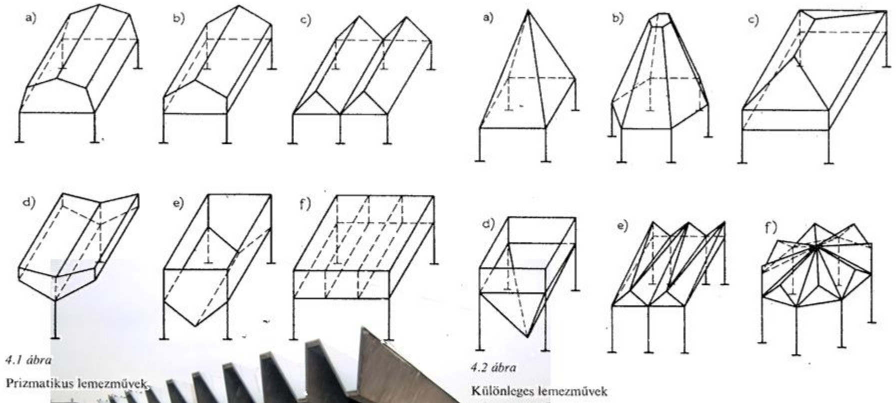
Sík felület — Görbült felület

tárcsa, lemez, lemezmű — egyszer g., kétszer g.

héj



Lemezművek



Héjak

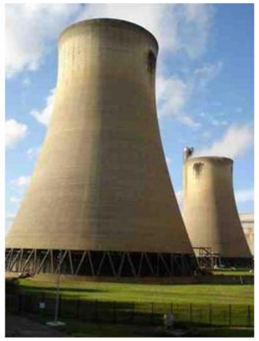
Brazil Nemzeti Múzeum, O. Niemeyer

MIT Kresge Auditorium – Boston (1955)

Palazzetto dello Sport, Róma, Nervi (1957)

Héjak

Heinz Isler (1961)



Oceonográfiai Múzeum. Valencia (F. Candela, 1997)

Héjak - Binishells

Dante Bini
Lakóépület, Ausztrália

Dante Bini
Monash University, Churchill, Ausztrália (1979)

Héjak - ellenpélda

Sydney Operaház,
J. Utzon dán építész (1973)

Nem biztosított a membránállapot kialakulása , mert nyúlásmenetes alakváltozásokra képes -> hajlított héj!
Előregyártott bordák és tetőpanelek!
Építés 10 évig tartott
költség 15-szeres!

Tárcsák

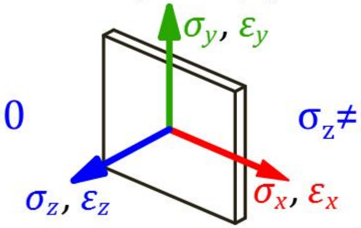
Síkbeli feszültségállapot

falszerkezetek

Síkbeli alakváltozás állapot

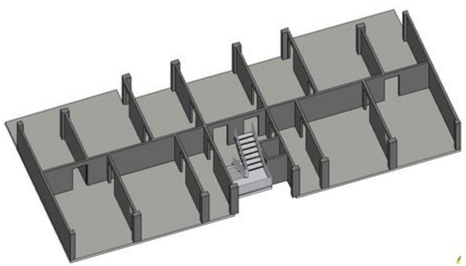
vonalas építmények metszete
(támfal, gátszerkezet, alagút stb.)

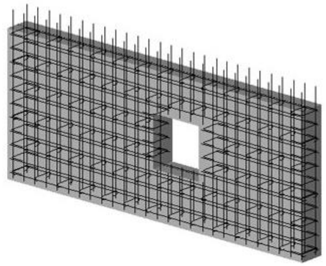
$\sigma_z=0, \varepsilon_z\neq0$



$\sigma_z\neq0, \varepsilon_z=0$

Tárcsák – síkbeli feszültség áll.

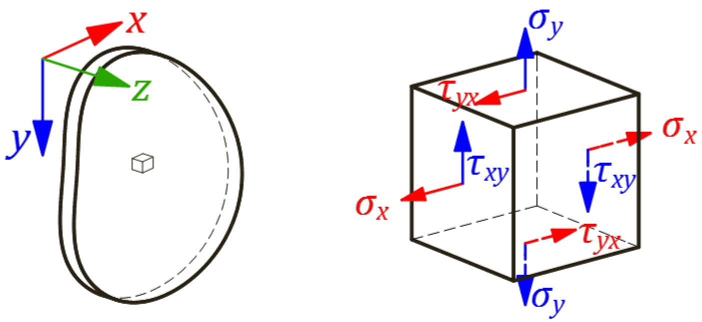




Alapfeltevések

- a tárcsa anyaga homogén, izotrop, lineárisan rugalmas
- a tárcsa vastagsága állandó
- a tárcsa vékony
- a középsík nem tér ki a síkjából
- a tárcsa kis elmozdulást végez, a középsík normálisán lévő pontok az alakváltozás után is a normálison maradnak
- síkbeli feszültségállapot
- elsőrendű elmélet
- a reakcióerők is csak a tárcsa síkjában hathatnak

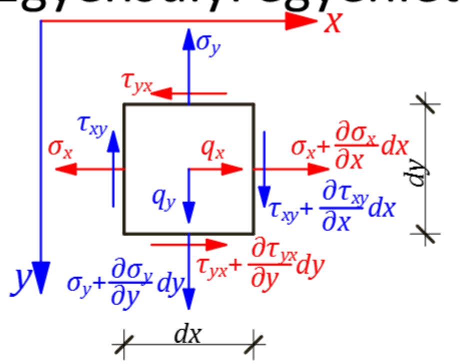
Egyensúlyi egyenlet



$$\mathbf{F}(x,y)=\begin{bmatrix}\sigma_x(x,y) & \tau_{xy}(x,y)\\ \tau_{yx}(x,y) & \sigma_y(x,y)\end{bmatrix} \qquad \mathbf{F}=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{yx} & \sigma_y\end{bmatrix}$$

Feltételezés: A feszültségek a vastagságok mentén állandóak.

Egyensúlyi egyenlet

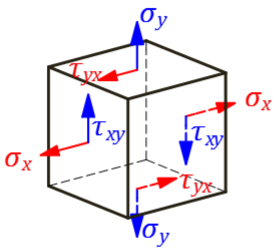


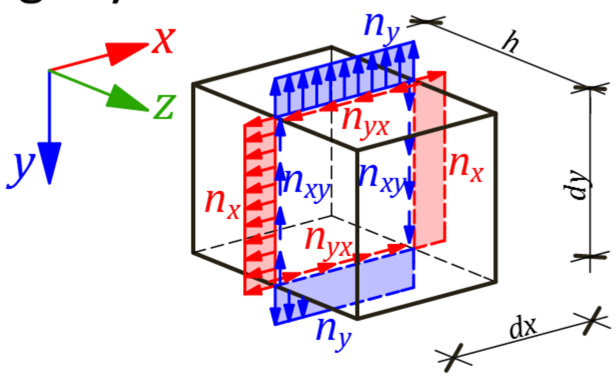
$$\sum F_{ix}: -\sigma_x\,dy\,h+\left(\sigma_x+\frac{\partial\sigma_x}{\partial x}dx\right)dy\,h-\tau_{yx}\,dx\,h+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}dy\right)dx\,h+q_x dy\,h=0$$

$$\sum F_{ix}: \frac{\partial\sigma_x}{\partial x}dxdy+\frac{\partial\tau_{yx}}{\partial y}dydx=0$$

$$\sum F_{ix}: \frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}=0 \qquad \sum F_{iy}: \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}=0 \qquad \tau_{xy}=\tau_{yx}$$

Igénybevételek





$n_x = n_x(x,y)$
$n_y = n_y(x,y)$
$n_{yx} = n_{yx}(x,y)$
$n_{yx} = n_{xy}$

$$n_x = \int_{-h/2}^{+h/2} \sigma_x \, dz = \sigma_x \int_{-h/2}^{+h/2} dz = \sigma_x \, h$$

$n_x = \sigma_x h \quad n_y = \sigma_y h \quad n_{yx} = \tau_{yx} h \quad n_{xy} = \tau_{xy} h \quad [kN/m]$

A feszültségállapot síkbeli!

Anyagegyenletek

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\, \sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\, \sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad G = \frac{E}{2(1+\nu)}$$

Az alakváltozás állapota térbeli!

Geometriai egyenletek

$$\varepsilon_x(x,y) = \frac{\partial u(x,y)}{\partial x}$$

$$\varepsilon_y(x,y) = \frac{\partial v(x,y)}{\partial y}$$

$$\gamma_{xy}(x,y) = \frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x}$$

$$\mathbf{A}(x,y) = \begin{bmatrix} \varepsilon_x(x,y) & \gamma_{xy}(x,y)/2 & 0 \\ \gamma_{xy}(x,y)/2 & \varepsilon_y(x,y) & 0 \\ 0 & 0 & \varepsilon_z(x,y) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & 0 \\ \gamma_{xy}/2 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

Ismeretlenek

$$\left.\begin{matrix} \sigma_x(x,y),\ \sigma_y(x,y),\ \tau_{xy}(x,y), \\ \varepsilon_x(x,y),\ \varepsilon_y(x,y),\ \varepsilon_z(x,y),\ \gamma_{xy}(x,y), \\ u(x,y),\ v(x,y) \end{matrix}\right\}\ 9 \text{ db}$$

Egyenletek

$$\left.\begin{matrix} \sum F_{ix}: \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} = 0 \qquad \sum F_{iy}: \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} = 0 \\ \varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\,\sigma_y) \quad \varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\,\sigma_x) \quad \varepsilon_z = -\dfrac{\nu}{E}(\sigma_x + \sigma_y) \quad \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\ \varepsilon_x(x,y) = \dfrac{\partial u(x,y)}{\partial x} \quad \varepsilon_y(x,y) = \dfrac{\partial v(x,y)}{\partial y} \quad \gamma_{xy}(x,y) = \dfrac{\partial u(x,y)}{\partial y} + \dfrac{\partial v(x,y)}{\partial x} \end{matrix}\right\}\ 9$$

+ peremfeltételek

Airy-féle feszültségfüggvények

$$\sigma_x(x,y) = \frac{\partial^2 F(x,y)}{\partial y^2}$$

$$\sigma_y(x,y) = \frac{\partial^2 F(x,y)}{\partial x^2}$$

$$\tau_{xy}(x,y) = -\frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Sir George Biddell Airy
királyi csillagász
1801-1892

(1863)

Tárcsaegyenlet

$$\sigma_x(x,y) = \frac{\partial^2 F(x,y)}{\partial y^2} \qquad \sigma_y(x,y) = \frac{\partial^2 F(x,y)}{\partial x^2} \qquad \tau_{xy}(x,y) = -\frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Egyensúlyi egyenletek:

$$\frac{\partial \sigma_x(x,y)}{\partial x} + \frac{\partial \tau_{yx}(x,y)}{\partial y} = \frac{\partial^3 F(x,y)}{\partial x \partial y^2} - \frac{\partial^3 F(x,y)}{\partial x \partial y^2} = 0$$

$$\frac{\partial \sigma_y(x,y)}{\partial y} + \frac{\partial \tau_{xy}(x,y)}{\partial x} = \frac{\partial^3 F(x,y)}{\partial x^2 \partial y} - \frac{\partial^3 F(x,y)}{\partial x^2 \partial y} = 0$$

Anyagegyenletek:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\,\sigma_y) = \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \nu\,\frac{\partial^2 F}{\partial x^2}\right)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\,\sigma_x) = \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu\,\frac{\partial^2 F}{\partial y^2}\right)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = -\frac{2(1+\nu)}{E}\frac{\partial^2 F}{\partial x \partial y}$$

Tárcsaegyenlet

$$\sigma_x(x,y) = \frac{\partial^2 F(x,y)}{\partial y^2} \qquad \sigma_y(x,y) = \frac{\partial^2 F(x,y)}{\partial x^2} \qquad \tau_{xy}(x,y) = -\frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Geometriai egyenletek: $\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$$\frac{\partial^2}{\partial x \partial y}\gamma_{xy} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^2}{\partial y^2}\varepsilon_x + \frac{\partial^2}{\partial x^2}\varepsilon_y$$

$$-\frac{\partial^2}{\partial x \partial y}\frac{2(1+\nu)}{E}\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2}{\partial y^2}\frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right)$$

$$-2(1+\nu)\frac{\partial^4 F}{\partial x^2 \partial y^2} = \frac{\partial^4 F}{\partial y^4} - \nu\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial x^4} - \nu\frac{\partial^4 F}{\partial x^2 \partial y^2}$$

$$\boxed{\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0}$$

Tárcsaegyenlet

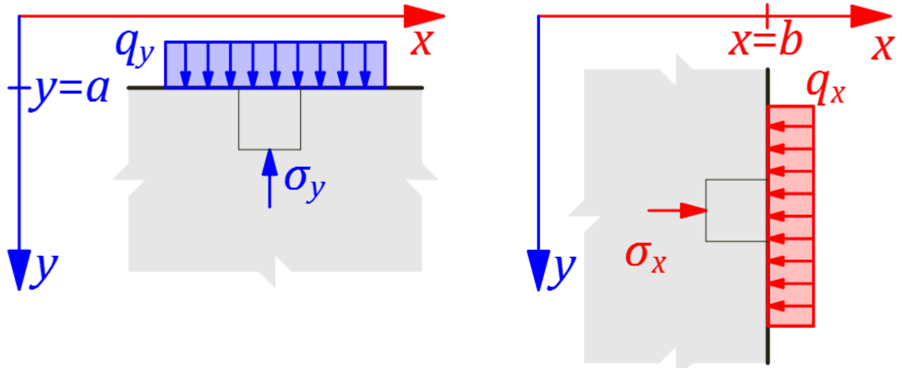
$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0$$

Harmonikus Laplace-operátor: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Biharmonikus differenciál egyenlet: $\Delta\Delta F = 0$

+ peremfeltételek

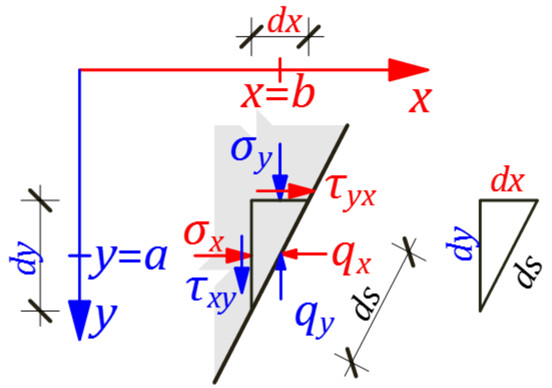
Peremfeltételek



$$\sigma_y(x,a) = \left.\frac{\partial^2 F(x,y)}{\partial x^2}\right|_{y=a} = -\frac{q_y}{h} \qquad \sigma_x(b,y) = \left.\frac{\partial^2 F(x,y)}{\partial y^2}\right|_{x=b} = -\frac{q_x}{h}$$

$$\tau_{xy}(x,a) = -\left.\frac{\partial^2 F(x,y)}{\partial x \partial y}\right|_{y=a} = 0 \qquad \tau_{xy}(b,y) = -\left.\frac{\partial^2 F(x,y)}{\partial x \partial y}\right|_{x=b} = 0$$

Peremfeltételek

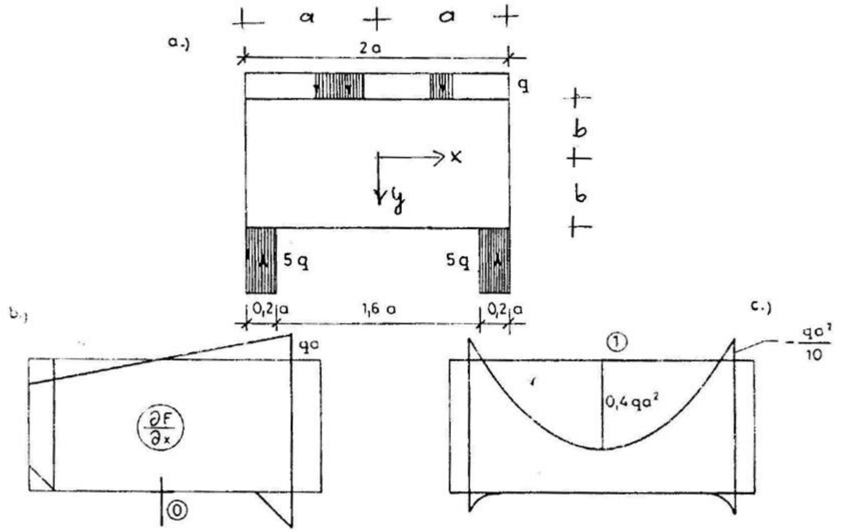


$$q_x \cdot ds = \sigma_x dy + \tau_{yx} dx \qquad q_y \cdot ds = \sigma_y dx + \tau_{xy} dy$$

$$q_x = \sigma_x \frac{dy}{ds} + \tau_{yx}\frac{dx}{ds} = \sigma_x \cos(\alpha) + \tau_{yx}\sin(\alpha)$$

$$q_y = \sigma_y \frac{dx}{ds} + \tau_{xy}\frac{dy}{ds} = \sigma_y \sin(\alpha) + \tau_{xy}\cos(\alpha)$$

Peremfeltételek



Polárkoordinátás alak

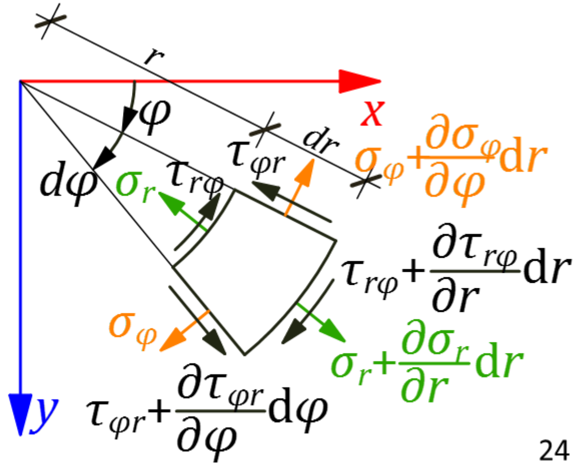
$$\Delta\Delta F(r,\varphi) = 0 \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{r}\frac{\partial F}{\partial r}\right) = 0$$

$$\sigma_r(r,\varphi) = \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{r}\frac{\partial F}{\partial r}$$

$$\sigma_\varphi(r,\varphi) = \frac{\partial^2 F}{\partial r^2}$$

$$\tau_{r\varphi}(r,\varphi) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \varphi}\right)$$



Polárkoordinátás alak

Körszimmetrikus geometria és teher esetén:

$$F(r,\varphi) \to F(r) \qquad \Delta() = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}()\right)$$

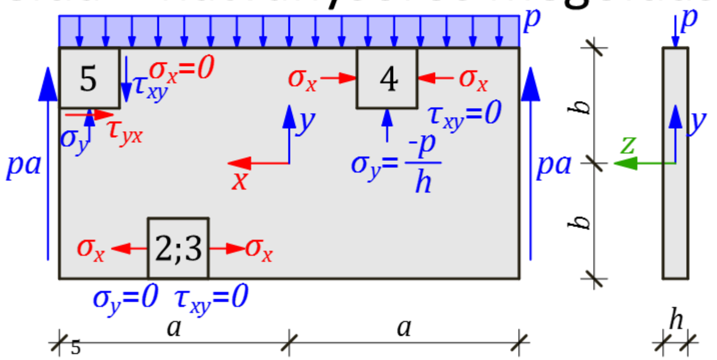
$$\frac{1}{r}\cdot\left[\frac{d}{dr}\left[r\cdot\left[\frac{d}{dr}(f(r))\right]\right]\right] \text{ simplify } \to \frac{d^2}{dr^2}f(r) + \frac{\frac{d}{dr}f(r)}{r}$$

$$\Delta\Delta F = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)\right)\right) = \boxed{\frac{\partial^4 F}{\partial r^4} + \frac{1}{r^3}\frac{\partial F}{\partial r} - \frac{1}{r^2}\frac{\partial^2 F}{\partial r^2} + \frac{2}{r}\frac{\partial^3 F}{\partial r^3} = 0}$$

$$\frac{1}{r}\cdot\left[\frac{d}{dr}\left[r\cdot\left[\frac{d}{dr}\left[\frac{1}{r}\cdot\left[\frac{d}{dr}\left[r\cdot\left(\frac{d}{dr}F(r)\right)\right]\right]\right]\right]\right]\right] \text{ simplify } \to \frac{d^4}{dr^4}F(r) + \frac{\frac{d}{dr}F(r)}{r^3} - \frac{\frac{d^2}{dr^2}F(r)}{r^2} + \frac{2\cdot\frac{d^3}{dr^3}F(r)}{r}$$

$$\sigma_r(r) = \frac{1}{r}\frac{\partial F}{\partial r} \qquad \sigma_\varphi(r) = \frac{\partial^2 F}{\partial r^2} \qquad \tau_{r\varphi} = 0$$

Példa – hatványsoros megoldás



$$F(x,y) = \sum_{i=1}^{5}\alpha_i f_i(x,y) = \alpha_1 y^5 + \alpha_2 y^3 + \alpha_3 x^2 y^3 + \alpha_4 x^2 y + \alpha_5 x^2$$

$$\sigma_x(x,y) = \frac{\partial^2 F(x,y)}{\partial y^2} = 20\alpha_1 y^3 + 6\alpha_2 y + 6\alpha_3 x^2 y$$

$$\sigma_y(x,y) = \frac{\partial^2 F(x,y)}{\partial x^2} = 2\alpha_3 y^3 + 2\alpha_4 y + 2\alpha_5$$

$$\tau_{xy}(x,y) = -\frac{\partial^2 F(x,y)}{\partial x \partial y} = -6\alpha_3 x y^2 - 2\alpha_4 x$$

Példa – hatványsoros megoldás

$$F(x,y) = \sum_{i=1}^{5}\alpha_i f_i(x,y) = \alpha_1 y^5 + \alpha_2 y^3 + \alpha_3 x^2 y^3 + \alpha_4 x^2 y + \alpha_5 x^2$$

$$\frac{\partial^4 F}{\partial x^4} = 0 \qquad \frac{\partial^4 F}{\partial x^2 \partial y^2} = 12\alpha_3 y \qquad \frac{\partial^4 F}{\partial y^4} = 120\alpha_1 y$$

1.

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 + 2\cdot 12\alpha_3 y + 120\alpha_1 y = 0 \qquad \boxed{\alpha_3 + 5\alpha_1 = 0}$$

$$\sigma_y(x,-b) = -2\alpha_3 b^3 - 2\alpha_4 b + 2\alpha_5 = \boxed{-\alpha_3 b^3 - \alpha_4 b + \alpha_5 = 0} \text{ 2.}$$

$$\tau_{xy}(x,-b) = -6\alpha_3 x b^2 - 2\alpha_4 x = \boxed{-3\alpha_3 b^2 - \alpha_4 = 0} \text{ 3.}$$

$$\sigma_y(x,b) = \boxed{2\alpha_3 b^3 + 2\alpha_4 b + 2\alpha_5 = -\frac{p}{h}} \text{ 4.}$$

5.

$$\sigma_x(a,b) = 20\alpha_1 b^3 + 6\alpha_2 b + 6\alpha_3 a^2 b = \boxed{10\alpha_1 b^2 + 3\alpha_2 + 3\alpha_3 a^2 = 0}$$

Példa – hatványsoros megoldás

$$\begin{bmatrix} 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & -b^3 & -b & 1 \\ 0 & 0 & -3b^2 & -1 & 0 \\ 0 & 0 & 2b^3 & 2b & 2 \\ 10b^2 & 3 & 3a^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -p/h \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & -b^3 & -b & 1 \\ 0 & 0 & -3\cdot b^2 & -1 & 0 \\ 0 & 0 & 2\cdot b^3 & 2\cdot b & 2 \\ 10\cdot b^2 & 3 & 3\cdot a^2 & 0 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-p}{h} \\ 0 \end{pmatrix} \to \begin{bmatrix} -\frac{p}{40\cdot b^3\cdot h} \\ -\frac{p\cdot(3\cdot a^2 - 2\cdot b^2)}{24\cdot b^3\cdot h} \\ \frac{p}{8\cdot b^3\cdot h} \\ -\frac{3\cdot p}{8\cdot b\cdot h} \\ -\frac{p}{4\cdot h} \end{bmatrix}$$

Példa – hatványsoros megoldás

$$F(x,y)=\sum_{i=1}^{5}\alpha_i f_i(x,y)=\alpha_1 y^5+\alpha_2 y^3+\alpha_3 x^2y^3+\alpha_4 x^2 y+\alpha_5 x^2=$$

$$=\frac{p}{4hb^3}\left(-\frac{1}{10}y^5+\left(\frac{b^2}{3}-\frac{a^2}{2}\right)y^3+\frac{1}{2}x^2y^3-\frac{3b^2}{2}x^2y-b^3x^2\right)$$

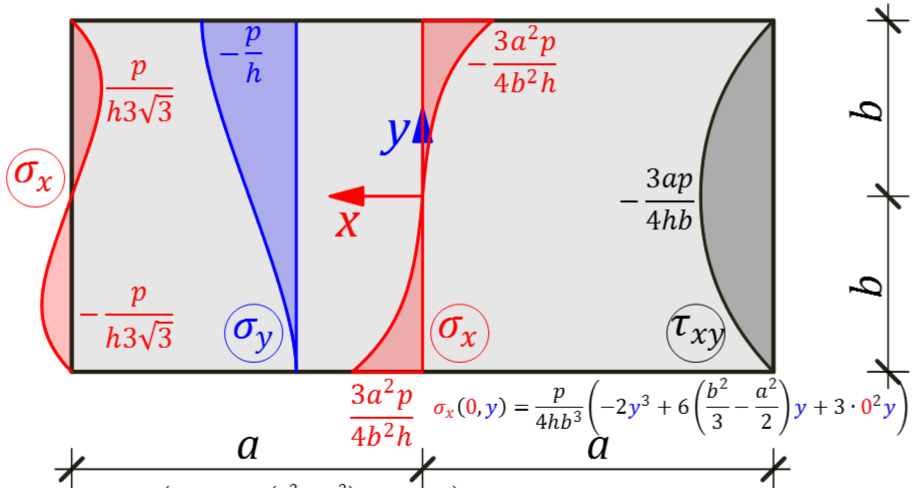
$$\sigma_x(x,y)=\frac{\partial^2F(x,y)}{\partial y^2}=\frac{p}{4hb^3}\left(-2y^3+6\left(\frac{b^2}{3}-\frac{a^2}{2}\right)y+3x^2y\right)$$

$$\sigma_y(x,y)=\frac{\partial^2F(x,y)}{\partial x^2}=\frac{p}{4hb^3}(y^3-3b^2y-2b^3)$$

$$\tau_{xy}(x,y)=-\frac{\partial^2F(x,y)}{\partial x\partial y}=\frac{p}{4hb^3}(-3xy^2+3b^2x)$$

Példa – hatványsoros megoldás

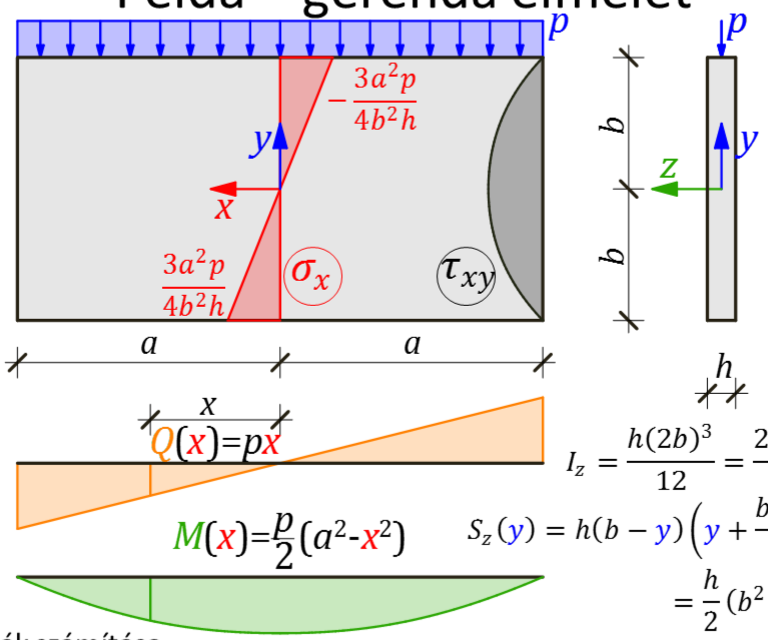
$$\sigma_y(x,y)=\frac{p}{4hb^3}(y^3-3b^2y-2b^3)\qquad \tau_{xy}(-a,y)=\frac{p}{4hb^3}(3ay^2-3b^2a)$$



$$\sigma_x(0,y)=\frac{p}{4hb^3}\left(-2y^3+6\left(\frac{b^2}{3}-\frac{a^2}{2}\right)y+3\cdot 0^2y\right)$$

$$\sigma_x(a,y)=\frac{p}{4hb^3}\left(-2y^3+6\left(\frac{b^2}{3}-\frac{a^2}{2}\right)y+3a^2y\right)$$

Példa – gerenda elmélet



$$I_z=\frac{h(2b)^3}{12}=\frac{2hb^3}{3}$$

$$S_z(y)=h(b-y)\left(y+\frac{b-y}{2}\right)=$$
$$=\frac{h}{2}(b^2-y^2)$$

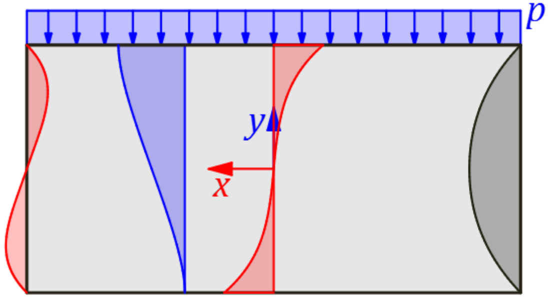
Példa – gerenda elmélet

$$\sigma_x(x,y)=\frac{M(x)}{I_z}(-y)=\frac{\frac{p}{2}(a^2-x^2)}{\frac{2hb^3}{3}}(-y)=\frac{-3p(a^2-x^2)}{4hb^3}y=\frac{p}{4hb^3}(-3a^2y+3x^2y)$$

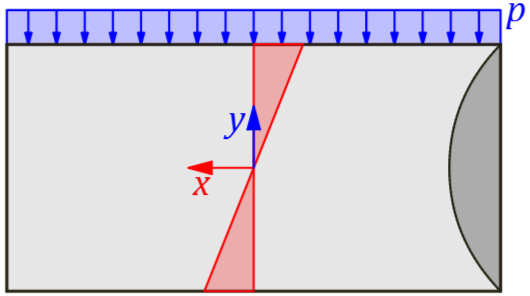
$$\sigma_y(x,y)=0$$

$$\tau_{xy}(x,y)=\frac{Q(x)S_z(y)}{I_z h}=\frac{px\frac{h}{2}(b^2-y^2)}{\frac{2hb^3}{3}h}=\frac{p}{4hb^3}(-3xy^2+3b^2x)$$

Példa – összehasonlítás



Tárcsaelmélet



Gerendaelmélet

Példa – összehasonlítás

Tárcsaelmélet	Gerendaelmélet
$\sigma_x(x,y) = \frac{p}{4hb^3}\left(-2y^3 + 6\left(\frac{b^2}{3} - \frac{a^2}{2}\right)y + 3x^2 y\right)$	$\sigma_x(x,y) = \frac{p}{4hb^3}(-3a^2 y + 3x^2 y)$
$\sigma_y(x,y) = \frac{p}{4hb^3}(y^3 - 3b^2 y - 2b^3)$	$\sigma_y(x,y) = 0$
$\tau_{xy}(x,y) = \frac{p}{4hb^3}(-3xy^2 + 3b^2 x)$	$\tau_{xy}(x,y) = \frac{p}{4hb^3}(-3xy^2 + 3b^2 x)$

VÉGE

Köszönöm a figyelmet!

Összeállította: Dr. Hortobágyi Zsolt
BME Tartószerkezetek Mechanikája TSZ