Introduction: what is it about?

Any book written on the introduction of *Statics* or *Dynamics* starts by positioning the subject within science: this approach is also adopted here but only a rough overview is given.

Mechanics is the branch of physics that deals with the conditions of rest and motion of bodies and materials. It can further be classified according to the consistency of the material (mechanics of gases, fluids and solids): it is only dealt with *solid bodies* here. A solid body can be considered either *perfectly rigid* (no deformation is possible) or *deformable* (even if the range of deformation is restricted). Conditions of motion and rest of bodies can be analysed without respect to their phisical reasons, using only phenomenological parameters like distance, velocity and acceleration: in theis cause we speak about *Kinematics*. If the reasons of rest or motions are also considered, the term *Kinetics* is applied. Finally, Statics and Dynamics are distinguished according to that the body is in rest or motion. (Note that there exist different concepts in sub- or coordination of these subjects, e.g. those considering Kinetics and Kinematics being two branches of Dynamics.)

On the scale of parameters we use in the description of motions, Newtonian mechanics is completely adequate, i.e., *Newton's laws* apply sufficiently:

1. *Principle of inertial motion.* "Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by *force* impressed".

2. *Law of Dynamics.* "The rate of change of momentum of an object is proportional to the resultant force acting on the body and is in the same direction", that is, the rate of change of *velocity* (*acceleration*, $\boldsymbol{a}$) of a material particle is proportional to the force $\boldsymbol{F}$ and the factor of proportion is the (constant) mass m of the body: $\boldsymbol{F}=m\,\boldsymbol{a}$.

3. *Principle of action and reaction.* "All forces occur in pairs, and these two forces are equal in magnitude and opposite in direction." It is important to emphasize that two forces in such a pair act on two different bodies in interaction. In other words, "every action has opposite and equalent reaction".

The first Newton's law should also be completed by stating that our *frame* (*coordinate system*) is *inertial*. It allows for a reformulation of this law as that any physical events proceed in the same way described in two frames moving with a constant relative velocity (that is, both for magnitude and direction) with respect to the other. For instance, the trajectory of a thrown ball is approximately parabolic either seen from the ground or through the window of a train travelling with uniform speed (the statement about parabolic shape will be proved later).

Frames have just been mentioned: even if there is no general rule for setting a coordinate system, computations require to set a frame either in a right-handed or left-handed fashion. Here we adopt the first choice: a right handed system is named after its property that axes x, y and z follow each other in the order of the first three fingers of our right hand (if all axes in such a system are perpendicular to each othar, we speak about a *Cartesian* system). Consequently, a customary planar coordinate system xy with x directed to the right and y upwards, z points in front of us. Another equivalent configuration is when x points to the right, z downwards and y in front of us (other combinations are still possible but these are the most common ones).

In Newton's second law, mass m is a *scalar*, that is, can be specified by a single number. $\boldsymbol{F}$ and $\boldsymbol{a}$, however, are *vectors* and have therefore both *magnitude* and *direction*. Such vector quantities are graphically displayed by arrows: an arrow has a tail and a tip that specify the *slope* of a vector. 'Slope' should not be confused with 'direction' of a vector, which is given by the slope and the sense (from the tail towards the tip) of a vector together.

Since vector operations will be used in this course extensively, let us give a short review on them.

Vectors

Vectors, as have already been mentioned, are directed: they have both magnitude and direction. In typography, vectors are typeset in bold (or bold italic) typeface for distinction (e.g., ***v*** can stand for velocity), but underline is more common in handwriting: $\underline{v}$ (note that overbar (arrow) can also be found as reference to vectors in both print or handwriting). In this workbook, bold typeface is applied in descriptive sections but it is replaced by underline in problems and examples in order to facilitate the comparison with hand-written solutions. In a graphical environment, vectors are represented by arrows.

A vector embedded in a coordinate system can be specified by its magnitude and direction. The magnitude is the length of the vector, treated commonly as a scalar variable (e.g., the magnitude of vector ***v*** is *v*), but it can also appear between absolute value signs in handwriting (that is, $v = |\underline{v}|$). The direction can be related to any other but fixed direction (e.g., coordinate axes) by setting ang angle as well. Another way of specification of vectors is possible as follows: perpendicular projections (called 'components' henceforth) of the vector to each coordinate axis are grouped into a column vector (delimited therefore by square brackets). Of course, a one-to-one correspondence should exist between these two ways of specification, since both of them result in the same and unique vector.

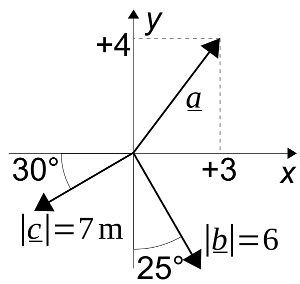
A vector of length 1 is called *unit vector*. Unit vectors along axes *x*, *y* and *z* are of primary importance and are denoted by symbols ***i***, ***j***, ***k***.

In *two dimensions*, any vector can uniquely be specified by two scalars: either by two coordinates of the tip of the vector started at the origin (see vector ***a*** in the following example) or by the length of the vector and the angle subtended by the vector and some of the coordinate axes (see vectors ***b*** or ***c*** of the same). There exists a conversion method between the two modes based on trigonometric functions and the Pythagorean theorem. (Recall that in a right triangle, a leg opposite to the angle over hypotenuse is the sine, adjacent leg over hypotenuse is the cosine, opposite leg over adjacent leg is the tangent of the angle.)

If a vector reflects any physical property, then each of its scalar components and its length has the same physical measure (unit).

Example 1

Specify vectors $\underline{a}, \underline{b}$ and $\underline{c}$.
Give the magnitude and direction of vector $\underline{a}$.



Solution

$$\underline{a}=\begin{bmatrix}3\\4\end{bmatrix}$$

$$\underline{b}=\begin{bmatrix}+6\cdot\sin 25°\\-6\cdot\cos 25°\end{bmatrix}=\begin{bmatrix}+2.536\\-5.438\end{bmatrix}$$

$$\underline{c}=\begin{bmatrix}-7\,\mathrm{m}\cdot\cos 30°\\-7\,\mathrm{m}\cdot\sin 30°\end{bmatrix}=\begin{bmatrix}-6.062\\-3.5\end{bmatrix}\mathrm{m}$$

magnitude of $\underline{a}$:

$$a=|\underline{a}|=\sqrt{3^2+4^2}=5$$

direction of $\underline{a}$

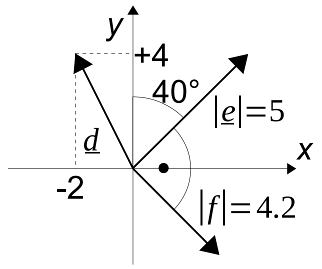
(angle subtended by $\underline{a}$ and $+x$):

$$\tan\alpha=\frac{|4|}{|3|}\rightarrow\alpha=53.13°$$

Positive or negative sign of any component is decided, to advantage, upon inspection; numeric values are then rounded off to four significant figures. One might ask why four and not three or five; some paragraphs were dedicated to this issue in the Introduction. Eventually, it is a matter of compromise between accuracy and reasonable efforts: experience of many years of education led us to adopt this convention. Although the number of figures influences the final precision but the precision itself can only be confidently evaluated via mathematical statistical methods. We should be familiar with that results will be nothing more precise if the 10th digit is put down, while the sign has already been mistyped.

Exercise 1

Specify vectors $\underline{d}, \underline{e}$ and $\underline{f}$.
Give the magnitude and direction of vector $\underline{d}$.



Solution

entries of $\underline{d}$ (order, sign):

$$\underline{d} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

entries of $\underline{e}$ (first by formula, then calculated and rounded):

$$\underline{e} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

entries of $\underline{f}$ (first by formula, then calculated and rounded):

$$\underline{f} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

magnitude of $\underline{d}$:

$$|\underline{d}| = \sqrt{\quad^2 + \quad^2} =$$

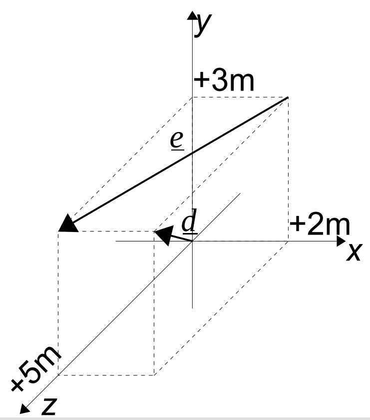
direction of $\underline{d}$ (angle measured from $+x$):

$$\tan \delta_{-x} = \frac{|\quad|}{|\quad|} = \qquad \rightarrow \delta_{-x} =$$

In the *space* any vector can be uniquely given by three scalars; if the tail of the vector is set to the origin, then, e.g., by three coordinates of its tip. Theoretically it would still possible to deal with magnitude and directions, where this latter property could be specified by angles α, β, γ subtended with axes *x, y, z*, respectively. Note that a magnitude and three angles would mean four parameters but, according to the Pythagorean theorem, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ should be satisfied.

Example 2

Specify vectors $\underline{d}$ and $\underline{e}$.
Give also three angles subtended by vector $\underline{d}$ and the coordinate axes, then check the result.



Solution

$\underline{d}=\begin{bmatrix}2\\3\\5\end{bmatrix}\text{m}$ magnitude of $\underline{d}$: $d=|\underline{d}|=\sqrt{2^2+3^2+5^2}=\sqrt{38}=6.164\,\text{m}$

$\cos\delta_x=\dfrac{2}{6.164}\rightarrow\delta_x=71.07°$

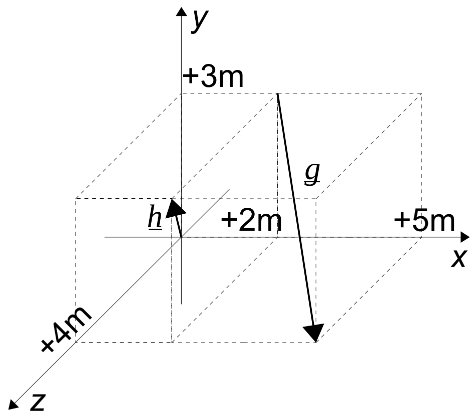
$\underline{e}=\begin{bmatrix}0-2\\3-3\\5-0\end{bmatrix}=\begin{bmatrix}-2\\0\\5\end{bmatrix}\text{m}$ $\cos\delta_y=\dfrac{3}{6.164}\rightarrow\delta_y=60.88°$

$\cos\delta_z=\dfrac{5}{6.164}\rightarrow\delta_z=35.79°$

$\therefore\cos^2 71.07°+\cos^2 60.88°+\cos^2 35.79°=1.000$ ✓

Exercise 2

Specify vectors $\underline{g}$ and $\underline{h}$.
Give also three angles subtended by vector $\underline{h}$ and the coordinate axes, then check the result.



Solution

entries of $\underline{g}$:

$$\underline{g}=\begin{bmatrix} \\ \\ \\ \end{bmatrix}=\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

entries of $\underline{h}$:

$$\underline{h}=\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

magnitude of $\underline{h}$:

$h=|\underline{h}|=$

$\cos\gamma_x=$

$\cos\gamma_y=$

$\cos\gamma_z=$

Check:

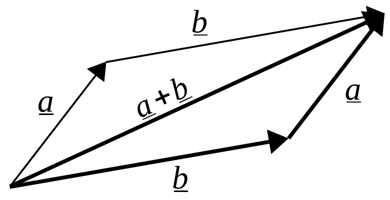
Sum and difference of vectors, precedence of operations

Vectors can be added both graphically and analytically. Graphical addition is made through concatenation of vectors in a tip-to-tail fashion (since the order of addition does not matter, the method is also known as the *parallelogram method*): in this case, one vector is followed by another such that the tip of the preceding and the tail of the following arrow coincide. Analytic addition of vectors is performed simply by addition of coordinates.

Example 3

Calculate vectors $\underline{a}+\underline{b}$, $\underline{b}+\underline{c}$ and $\underline{a}+\underline{b}+\underline{c}$

if $\underline{a}=\begin{bmatrix}0.8\\1.2\\0.5\end{bmatrix}, \underline{b}=\begin{bmatrix}-0.8\\0.4\\-0.7\end{bmatrix}, \underline{c}=\begin{bmatrix}0.9\\-1.1\\1.1\end{bmatrix}$!



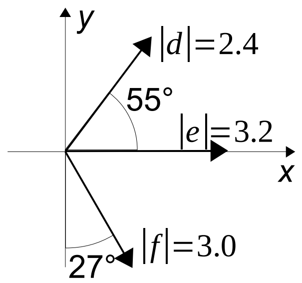
Solution

$$\underline{a}+\underline{b}=\begin{bmatrix}0.8\\1.2\\0.5\end{bmatrix}+\begin{bmatrix}-0.8\\0.4\\-0.7\end{bmatrix}=\begin{bmatrix}0.8+(-0.8)\\1.2+0.4\\0.5+(-0.7)\end{bmatrix}=\begin{bmatrix}0.0\\1.6\\-0.2\end{bmatrix}$$

$$\underline{b}+\underline{c}=\begin{bmatrix}-0.8+1.1\\0.4+(-1.1)\\-0.7+1.1\end{bmatrix}=\begin{bmatrix}0.1\\-0.7\\0.4\end{bmatrix} \qquad \underline{a}+\underline{b}+\underline{c}=\begin{bmatrix}0.8+(-0.8)+0.9\\1.2+0.4-(1.1)\\0.5+(-0.7)+1.1\end{bmatrix}=\begin{bmatrix}0.9\\0.5\\0.9\end{bmatrix}$$

Exercise 3

Calculate vectors $\underline{d}+\underline{e}$, $\underline{e}+\underline{f}$ and $\underline{f}+\underline{d}+\underline{e}$ if magnitudes and directions are given according to the figure



Solution

$$\underline{d}+\underline{e}=\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

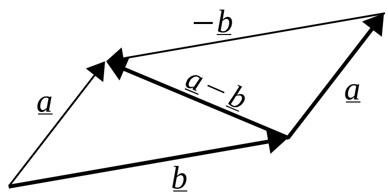
$$\underline{e}+\underline{f}=\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\underline{f}+\underline{d}+\underline{e}=\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Notice that ***a***+***b***=***b***+***a*** . Multiplication by a scalar can be introduced by the successive addition of the vector itself, where the resultant vector $\alpha \boldsymbol{v}$ will be parallel to ***v*** ;depending on the scalar *α*, the vector is either stretched, $(|\alpha|>1)$, shortened $(|\alpha|<1)$, or even mirrored $(\alpha<0)$.

A vector multiplied by -1 is called the negative of that vector: $\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}$.

The difference of two vectors is denoted by the symbol ***a***-***b***, and ***a***-***b*** = ***a***+(-***b***). The order of subtraction is not reversible: ***a***-***b*** = -(***b***-***a***).



Example 4

Calculate vectors $\underline{a}-\underline{b},\ \underline{b}-\underline{a}$

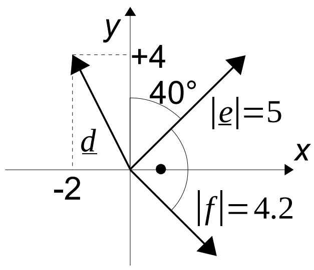
if $\underline{a}=\begin{bmatrix}12.5\\-5.4\\8.73\end{bmatrix},\underline{b}=\begin{bmatrix}3.9\\-11.3\\6.6\end{bmatrix}$.

Solution

$$\underline{a}-\underline{b}=\begin{bmatrix}12.5-3.9\\-5.4-(-11.3)\\8.73-6.6\end{bmatrix}=\begin{bmatrix}8.6\\5.9\\2.13\end{bmatrix}\qquad \underline{b}-\underline{a}=\begin{bmatrix}3.9-12.5\\-11.3-(-5.4)\\6.6-8.73\end{bmatrix}=\begin{bmatrix}-8.6\\-5.9\\-2.13\end{bmatrix}$$

Exercise 4

Calculate vectors $\underline{e}-\underline{d},\ \underline{d}-\underline{e}$
based on the data of Exercise 1.



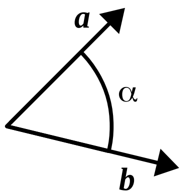
Solution

$$\underline{e}-\underline{d}=\begin{bmatrix}\quad\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}$$

$$\underline{d}-\underline{e}=\begin{bmatrix}\quad\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}$$

Projection of a vector onto another; scalar product

Scalar product (or dot product, ***a·b***, named exactly after the dot sign of multiplication) of two vectors is a scalar number, proportional both to the length of each vector and the cosine of the angle enclosed by them. Since this angle must be between 0° and 180°, its cosine is between −1 and 1. In extremal positions the two vectors either point in the same ($\cos\alpha = 1$) direction or ($\cos\alpha = -1$) in opposite directions.



$$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\alpha$$

$$0°\le\alpha\le180° \rightarrow -1\le\cos\alpha\le1$$

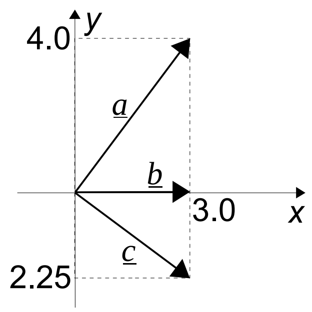
It can also be computed as the sum of products of vector coordinates as follows:

$$\boldsymbol{a}\cdot\boldsymbol{b}=\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix}\cdot\begin{bmatrix}b_x\\b_y\\b_z\end{bmatrix}=a_xb_x+a_yb_y+a_zb_z\quad .$$

The two methods above are, of course, equivalent.

Example 5

Calculate dot products of vectors
$\underline{a}, \underline{b}$ and $\underline{c}$ pairwise.
Calculate the angles enclosed by the vectors as well.



Solution

$\underline{a}\cdot\underline{b}=3\cdot3+4\cdot0=9$

$\underline{a}\cdot\underline{c}=3\cdot3+4\cdot(-2.25)=0$

$\underline{b}\cdot\underline{c}=3\cdot3+0\cdot(-2.25)=9$

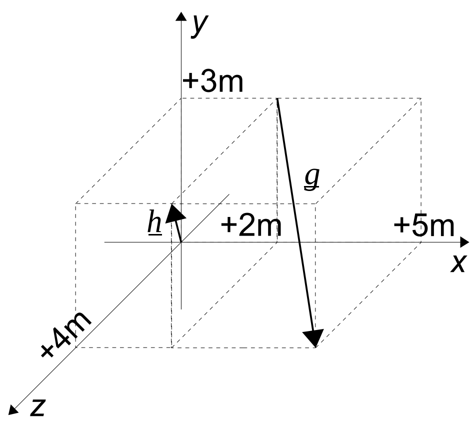
$$\cos\alpha_{ab}=\frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}=\frac{9}{\sqrt{3^2+4^2}\sqrt{3^2+0^2}}\rightarrow\alpha_{ab}=53.13°$$

$$\cos\alpha_{ac}=\frac{\underline{a}\cdot\underline{c}}{|\underline{a}||\underline{c}|}=\frac{0}{\sqrt{3^2+4^2}\sqrt{3^2+2.25^2}}\rightarrow\alpha_{ab}=90°$$

$$\cos\alpha_{bc}=\frac{\underline{b}\cdot\underline{c}}{|\underline{b}||\underline{c}|}=\frac{9}{\sqrt{3^2+0^2}\sqrt{3^2+2.25^2}}\rightarrow\alpha_{ab}=36.87°$$

Exercise 5

Calculate dot products of vectors
$\underline{g}$ and $\underline{h}$ of Exercise 2.
Calculate the angles enclosed by the vectors as well.



Solution

$$\underline{g}\cdot\underline{h}=$$

$$\cos\alpha_{gh}=\frac{\underline{g}\cdot\underline{h}}{|\underline{g}||\underline{h}|}=$$

Some special cases are worth noting here:

- If one vector in a dot product is a unit vector, then the product itself corresponds to the (signed) projection of the other vector onto the direction of the unit vector.
- Some other special cases depend on angles between vectors:
  - dot product of two perpendicular vectors equals zero,
  - dot product of two parallel (unidirectional) vectors equals the product of their lengths,
  - dot product of two parallel but oppositely directed vectors equals the negative of product of their lengths.

Kinematics of a material particle

Definition: Material particle is interpreted as a body without any direction or extension whose position can uniquely be defined by a single position vector.

Kinematics of a material particle is dedicated to the determination of parameters of motion of that particle: velocity and acceleration as functions of the position, or, velocity and position as functions of the acceleration are calculated here.

Rectilinear motion

Rectilinear motion can occur along any straight line. (The position of the particle could be given by some function $\boldsymbol{r}(t)=\boldsymbol{r}_0+\boldsymbol{e}_\tau \cdot s(t)$ of *t*, where $\boldsymbol{r}_0$ is an initial position vector, and $\boldsymbol{e}_\tau$ is a unit vector along the line of motion.) But the coordinate system can be set for convenience such that the particle moves along axis *x* (or this could either be *y* but we adopt the previous choice here).

In this case, the position of the particle is given by the *position function* $x(t)$.

The measure of change of position function in time (in mathematics, the time derivative of the function) is called *velocity function*: $v(t)$.

The measure of change of velocity function in time (in mathematics, the time derivative of the function) is called *acceleration function*: $v(t)$.

(Combining the mathematical definitions above, acceleration is the second time derivative of the position function.)

All the above measures are signed, i.e., if they are positive, they 'point ahead', if negatives, they 'point back'.

Definition: A rectilinear motion is uniformly accelerating if the acceleration is time-independent.

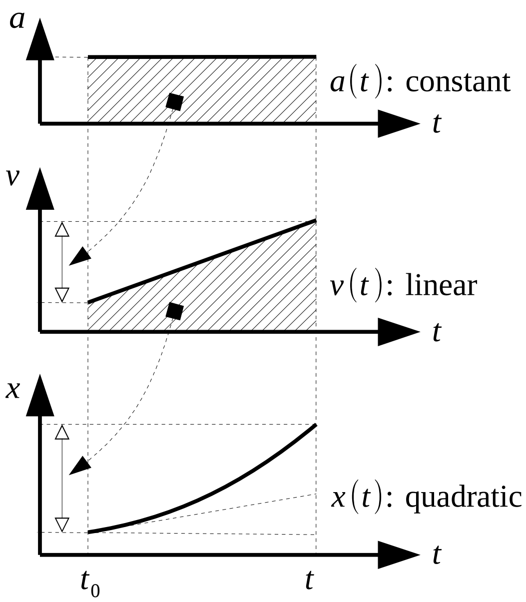
For such a case, the following formulae apply:

$$x(t)=\frac{a}{2}(t-t_0)^2+v_0\cdot(t-t_0)+x_0$$

$$v(t)=a\cdot(t-t_0)+v_0$$

$$x(t)=x_0+\frac{v_0+v(t)}{2}\cdot(t-t_0)$$

$$v^2(t)=v_0^2+2\cdot a\cdot(x(t)-x_0)$$



In addition to the formal knowledge, it is worth knowing which parameter does *not* appear in each formula (current velocity in the first, current position in the second, acceleration in the third and time in the fourth one): an equation will be written to advantage by excluding any variables that are neither known or sought for in the given problem.

Four above formulae can also be extracted from three plots on the right hand side, proceeding downwards, integration makes all areas to be proportional to the change of values displayed just above them.

The first formula shows the cumulative effect of constant initial value (position), initial velocity and the acceleration.

The second formula concerns the effect of initial velocity and the acceleration.

The third one calculates the distance from an average velocity.

The latter one is a consequence of the first and second: elimination of time leads to a direct link between velocity and distance.

All these formulae can further be simplified by an appropriate choice of the coordinate system. It is appropriate to set initial position and time such that $x_0=0$ and $t_0=0$ is satisfied. In this case, formulae read as follows:

$$x=\frac{a}{2}t^2+v_0 t=\frac{v_0+v}{2}t,\quad v=v_0+at,\quad v^2=v_0^2+2ax$$

Unfortunately, these formulae must be known by heart (like short poems...).

Uniform rectilinear motion

A special case of rectilinear motion is obtained when $a=0$. This is called uniform rectilinear motion; in such case, the velocity is constant: $v=v_0$ and $x(t)=x_0+v_0\cdot t$, which formulae can also be distilled from those of the uniformly accelerating motion.

Example 6

Let the velocity of a material particle be given as $v(t)=20-4\cdot t$ (m/s, if the unit of time is s).

Determine the acceleration of the particle.

Determine the distance from the initial position of the particle after 8 seconds elapsed.

At which position it reaches the velocity of 10 m/s?

Solution

With $t_0=0$, the parameters can be written into the function of velocity as follows:

$$v_0+a\cdot t=20+(-4)\cdot t \quad\rightarrow\quad v_0=+20\,\text{m/s}, a=-4\,\text{m/s}^2$$

The acceleration is therefore $\boxed{-4\,\text{m/s}^2}$.

The current position of the particle is sought for with respect to the initial configuration, so the origin can be fixed at $x_0=0$. With the use of the first formula we get:

$$x(t)=\frac{-4}{2}t^2+20\cdot t \;\rightarrow x(8)=\frac{-4}{2}8^2+20\cdot 8\rightarrow \quad \boxed{x(8)=32\,\text{m}}$$

The last question can be reformulated, asking when the particle reaches the velocity of $10\,\text{m/s}$ and where it will be then. With this approach, the answer is as follows:

$$v(t)=v_0+a\cdot t\rightarrow 10=20-4\cdot t\rightarrow t=2.5\,\text{s}\quad ,$$

that yields the position as:

$$x(t)=\frac{-4}{2}t^2+20\cdot t \;\rightarrow x(2.5)=\frac{-4}{2}2.5^2+20\cdot 2.5\rightarrow \quad \boxed{x=37.5\,\text{m}}$$

A more direct answer can be given for the same question using information on initial and final velocities, as well as the acceleration, hence the fourth formula contains only one unknown:

$$v^2=v_0^2+2\cdot a\cdot x\rightarrow 10^2=20^2+2\cdot(-4)\cdot x\rightarrow \quad \boxed{x=37.5\,\text{m}}$$

Exercise 6

Let the velocity of a material particle be given as $v(t)=6\cdot t-3$ (m/s, ha az időt s-ban mérjük).

Determine the acceleration of the particle.

Determine when the particle gets to a distance of 15 m with respect to its initial position at *t*=0.

Where does it reach the velocity of 18 m/s (with respect to its initial position)?

Solution

The essential parameters of motion can be read from the velocity function:

$v(t)=6\cdot t-3 \quad \rightarrow v_0= \qquad , \; a=$ (Mind the signs and units.)

In the second question, time, position, initial velocity and acceleration are either asked or known but there is nothing to do with current velocity. Therefore, the appropriate formula is as follows:

$=$ that can be ordered to zero to get a quadratic equation:

$0=$. Its solution is

$t_{1,2}=$

Among the two values obtained, only one has a real physical interpretation: $t=$

In order to answer the third question, let us look for the formula which contains initial and current (final) velocity, acceleration and position but not the time. Let us plug in all known values to have

, which yields: $x=$

Example 7

Two vehicles are travelling behind each other at a distance of 15 m, both have an initial velocity of ($v_0=60\,\text{km/h}$) . The one in front brakes and stops after an uniform deceleration of $2\,\text{m/s}^2$. The other starts braking one second later and tries to stop with the same deceleration.

Will the vehicles crash? If yes, then specify when, where and what is the difference of velocities at the time of crashing.

Solution

The vehicles crash if there is an instant when their position coincide. It can happen either when both are in progress or when the first one is already stopped. Let us assume that the first scenario gets realized.

Let us write the position of the leading vehicle as a function of the distance covered if the origin and the initial instant are set to the place and time where and when it starts to brake, respectively:

$$x_1(t)=\frac{a}{2}t^2+v_0\cdot t=\frac{-2}{2}t^2+\frac{60}{3.6}\cdot t=-t^2+16.67\,t$$

The position of the trailing vehicle at the instant $t=0$ is -15 , and the vehicle travels at a constant velocity through 1 second. Its position when starts braking can be written as

$x_{2,0}=-15+16.67\cdot 1=+1.67\,\mathrm{m}$.

With this in mind, the position of the vehicle as a function of the time in the braking period reads:

$$x_2(t)=\frac{a}{2}(t-t_0)^2+v_0\cdot(t-t_0)+x_{2.0}=\frac{-2}{2}(t-1)^2+16.67\cdot(t-1)+1.67\ ,$$

after simplification we get: $x_2(t)=-t^2+18.67\cdot t-16$. Equating this with the position of the other one: $16.67\cdot t-t^2=-16+18.67\cdot t-t^2 \rightarrow \quad t=8\,\mathrm{s}$.

Let us verify now our initial assumption: the velocity of the leading and trailing vehicles are $v_1(8)=16.67-2\cdot 8=+0.67\,\mathrm{m/s}$ and $v_2(8)=16.67-1\cdot(8-1)=+2.67\,\mathrm{m/s}$, respectively. This means that none of them is fully stopped, more precisely, both of them still proceed *ahead*. (According to our equations, both vehicles should immediately start backwards after an instantaneous stop; therefore, a negative velocity would mean a crash of a reversing vehicle, which do not match with the problem statement. In such a case, another question should rather be asked: could the trailing vehicle reach the position where the other one stopped?) The answer is definitely yes, they will have crashed in 8 seconds after the leading one starts braking. The difference of velocities at the crash is $+2.67-0.67=2\,\mathrm{m/s}$, the crash occurs at $x_1(8)=16.67\cdot 8-8^2=$ $\boxed{69.36\,\mathrm{m}}$ measured from the point where the first braking happens.

Exercise 7

Two vehicles start from rest (in the same direction) on a drag race. The first one starts 0,5 seconds earlier, its acceleration is $a_1=3\,\mathrm{m/s^2}$. The second one accelerates by $a_2=3.2\,\mathrm{m/s^2}$.

Calculate the advantage of the first car (both for velocity and distance).

When and where the second car hits the first one?

Solution

Let us express the position of both cars in the function of time such that position is measured from the startline and time is measured from the start of the second car. Thus, the time when the two functions are equal-valued correspondsto th instant when they meet again.

Motion of the first car:

In addition to the advantage gained by the early starting, it is necessary to know the velocity and position of the first car when the other starts:

$v_{1,0}=$, $x_{1,0}=$

If time *t* starts together with the second car, how the position of the first car can be expressed interms of *t*?

$$x_1(t)=\frac{a_1}{2}t^2+v_{1,0}\cdot t+x_{1,0}=$$

Motion of the second car

Because of the previous setting of initial position and time,

$$x_2(t)=$$

When they meet

A time instant *t* is sought for when $x_1=x_2$:

Let us order the above equation to 0: $0=$, whose solution is:

$$t_{1,2}=$$

Which of them has a physical relevance? $t=$

Where will the cars be then?

Example 8

Positions of a uniformly accelerating particle are $1\,\text{m},\ 4\,\text{m},\ 10\,\text{m}$ at time instants $t=0\,\text{s},\ t=2\,\text{s},\ t=4\,\text{s}$, respectively.

Write the position of the particle as a function of time.

Solution

Because of the uniform acceleration, parameters $a, v_{0,} x_0$ of the function $x(t)=\frac{a}{2}\cdot t^2+v_0\cdot t+x_0$ should be determined.

From $t=0$ it follows that $x_0=1\,\text{m}$.

In the first and next two-second period, the average velocity of the particle was $\frac{4-1}{2-0}=1.5\,\text{m/s}$, and $\frac{10-4}{4-2}=3\,\text{m/s}$, respectively. These averages are also the current velocity measured in the middle of each time period, at time instances $t=\frac{2-0}{2}=1\,\text{s}$ and $t=\frac{4-2}{2}=3\,\text{s}$. Writing them into the formula $v(t)=v_0+a\cdot t$, a system of two equations $\begin{matrix} 1{,}5=v_0+a\cdot 1 \\ 3{,}0=v_0+a\cdot 3 \end{matrix}$ is generated. Subtracting these equations one from another, the acceleration is obtained first, then the initial velocity can be calculated as follows: $\begin{matrix} a=0.75\,\text{m/s}^2 \\ v_0=0.75\,\text{m/s} \end{matrix}$.

Thus, the position of the particle is given by $\boxed{x(t)=0.375\cdot t^2+0.75\cdot t+1.0\,[\text{m}]}$.

Exercise 8

Velocities of a uniformly accelerating particle are $2\,\mathrm{m/s}$, $4\,\mathrm{m/s}$ at positions $x=1\,\mathrm{m}$, $x=3\,\mathrm{m}$, respectively.

Calculate the velocity of the particle at the position $x=5\,\mathrm{m}$.

Solution

Both initial and final velogities, as well as the length is given for the interval. From these data, acceleration and time required to cover the given distance can be obtained from separate equations.

Calculating acceleration

Let us consider the formula not including time parameter as follows:

With reference to this acceleration we have (at x = 5 m/s)

Solution 2 (through calculating time)

In order to get the duration of the first interval, the formula of distance as a function of average velocity can be used:

Now, the acceleration as the rate of change of velocity reads:

The time when the third point is reached can be deduced from the formula below:

, if ordered to zero we have:

$t_{1,2}=$

The only time of physical relevance is $t=$,

from which the velocity can be got; $v=$ → $v=$

Example 9

An inextensional body (particle) travels by a uniform rectilinear motion such that it covers 8 metres in 10 seconds. Calculate the time required for the body to get to 43 metres from its point of departure.

Solution

The (constant) velocity is obtained from the first condition as follows: $8 = v \cdot 10 \rightarrow v = 0.8\,\text{m/s}$

From this, the covered distance and the time is $x(t) = 0.8 \cdot t \rightarrow 43 = 0.8 \cdot t \rightarrow \boxed{t = 53.75\,\text{m}}$.

Exercise 9

An inextensional body (particle) travels by a uniform rectilinear motion such that it covers 10 metres in 8 seconds. Calculate the time required for the body to get to 34 metres from its point of departure.

Solution

Writing given numbers into the canonical form of position function we get

$$x(t) = x_0 + v \cdot t \rightarrow$$

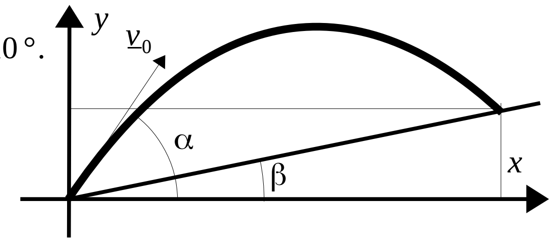
The position in the time in question yields the time itself as

Kinematics of a general motion in a plane

Prpendicular motions within the plane of motion can be dealt with as independent rectilinear motions. This principle can also be extended to three dimensions but such 3D problems are not discussed here. If the motion of a thrown body is analysed without considering the air resistance, the body will accelerate downwards vertically due to gravity (its sign depends on the direction of the vertical coordinate axis), while its horizontal velocity component is constant.

Example 1

An archer stands at the bottom of an incline of $\beta=10°$. He launches an arrow under the angle of $\alpha=40°$ with an initial velocity of $v_0=90\,\text{km/h}$. Find the position where the arrow hits the ground (air resistance is neglected).



Solution

Without air resistance, the acceleration of the body is exactly $g=9{,}81\,\text{m/s}^2$ downwards. With respect to the coordinate directions, accelerations are as follows:

$$a_x=0\,\text{m/s}^2, a_y=-9.81\,\text{m/s}^2.$$

In other words, there is a horizontal uniform rectilinear and a vertical uniformly accelerating rectilinear motion: $x(t)=x_0+v_x\cdot t,\ y(t)=y_0+v_{0y}\cdot t+\frac{a_y}{2}\cdot t^2.$

With respect to the coordinate system again, $x_0=y_0=0$.

Projections (components) of the initial velocity along *x* and *y*, respectively:

$$v_x=v_0\cdot\cos\alpha=\frac{90}{3.6}\cdot\cos 40°=19.15\,\text{m/s},$$

$$v_{0y}=v_0\cdot\sin\alpha=\frac{90}{3.6}\cdot\cos 40°=16.07\,\text{m/s}.$$

Thus, the positions as a function of time are as follows:

$$x(t)=19.15\cdot t,\quad y(t)=16.07\cdot t-4.905\cdot t^2.$$

When the arrow hits the ground, $y(t)=x(t)\cdot\tan\beta$, which can be expressed in terns of time:

$$16.07\cdot t-4.905\cdot t^2=19.15\cdot t\cdot\tan 10° \quad\rightarrow\quad 12.69\cdot t-4.905\,t^2=0.$$

There are two solutions: $t=0\,\text{s}$ and $t=2.587\,\text{s}$, since the trajectory of the arch crosses the line of the ground in two points. The first time instant indicates when the arrow was launched, the other one is important in the solution. The position of the arrow when hitting the ground is

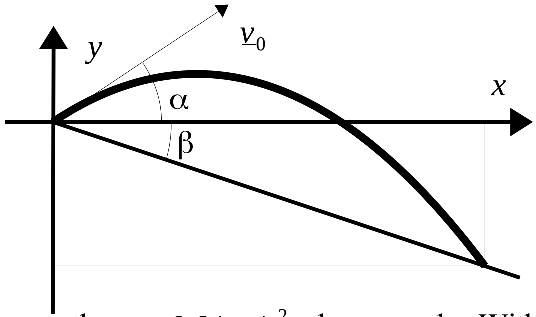
$$x(2.587)=19.15\cdot 2.587\rightarrow \qquad \boxed{x=49.54\,\text{m}}$$

$$y(2.587)=16.07\cdot 2.587-4.905\cdot 2.587^2\rightarrow \qquad \boxed{y=8.746\,\text{m}}$$

Remark:

The time elapsed from the instant of launching can be expressed with the horizontal position: $t=x/19.15$, which can further be written into the function of the vertical position. The resultant expression, $y=\frac{16.07}{19.15}\cdot x-\frac{4.905}{19.15^2}x^2$ corresponds to a second-order polynomial; therefore, the trajectory of the arch is parabolic.

Exercise 1



An archer stands on the top of an incline of $\beta=10°$.
He launches an arrow under the angle of $\alpha=15°$ with an initial velocity of $v_0=100\,\text{km/h}$.
Find the position where the arrow hits the ground (air resistance is neglected).

Solution

Without air resistance, the acceleration of the body is exactly $g=9{,}81\,\text{m/s}^2$ downwards. With respect to the coordinate directions, accelerations are as follows:

$$a_x=\ldots\ldots\ldots\,\text{m/s}^2\ ,\quad a_y=\ldots\ldots\ldots\,\text{m/s}^2$$ (mind the signs)

Now stop for a while and characterize the motion of the arrow:

horizontally: ....

vertically: ....

Parametric form of coordinates as a function of time:

$$x(t)=$$
$$y(t)=$$

where unknown parameters are x_0, v_{0x}, y_0, v_{0y} .

Due to the settings of the coordinate system, $x_0=\ldots\ldots\ldots\ldots\ldots,\ y_0=\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Initial velocity: $v_0=\dfrac{100}{3{,}6}=\ldots\ldots\ldots\ldots\ldots\,\text{m/s}$, whose (signed) components are

$$v_{0x}=..27.78\cdot\ldots.15°=\ldots\ldots\ldots.\,\text{m/s}\ ,$$
$$v_{0y}=..27.78\cdot\ldots.15°=\ldots\ldots\ldots.\,\text{m/s}\ .$$

Numeric expression of the coordinates as a function of time are as follows:

$$x(t)=\ldots\ldots.\cdot t$$
$$y(t)=\ldots\ldots\ldots.\cdot t^2\ ..\ \ \ldots\ldots\ldots\cdot t$$

When the arrow hits the ground: $y(t)=\ldots\ldots\ldots\ldots\ldots.\cdot x(t)$. Plugging the two above coordinate functions in: $\ldots\ldots\ldots..\cdot t^2\ ..\ \ \ldots\ldots\ldots\cdot t=\ldots\ldots\ldots\ldots.\cdot\ldots\ldots\ldots\ldots.\cdot t$. Let us solve it for *t*: $t_{1,2}=\vdots$

What does the solution $t=0$ mean? ....

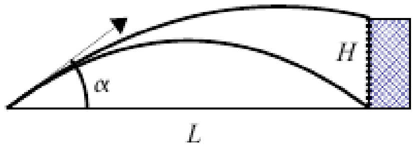
Has the nontrivial solution any physical relevance?.........................

If yes, where is the arrow then?

$$x(......) = ...\cdot......... = ...\ \text{m}$$
$$y(.......) = ...\cdot......^2\ ....\cdot....... = ...\ \text{m}$$

Example 2

Soccer fans shoot for goal in the break of the match from the midfield: the goal of height $H = 2.44$ stands at a distance of $L = 45\,\text{m}$.
One of the competitors kicks specially such that the velocity vector of the ball upon kicking encloses $\alpha = 40°$ with the plane of the pitch.
How great the initial velocity of kicking should be in order that the ball gets into the goal without a bounce?



Solution

If a coordinate system, with axis *x* pointing right and *y* up, is set to the point of kicking, then we can look for an initial velocity for which x = L occurs simultaneously with a *y* between zero and *H*. For this purpose, time can be expressed as a function of the initial velocity, then it is plugged in the function of height, for which two inequalities must hold.

Because of the neglected air resistance, the ball has no horizontal acceleration; itt accelerates vertically downwards by *g*. Taking the initial velocity of the ball and the settingas of the coordinate system into account, both coordinates depend on time as follows:

$$x(t) = v_0 \cdot \cos 40° \cdot t$$

$$y(t) = -\frac{g}{2} \cdot t^2 + v_0 \cdot \sin 40° \cdot t$$

The time elapsed until the ball reaches the vertical plane of the goal is obtained from the first equation:

$$t = \frac{L}{v_0 \cdot \cos 40°},$$

the vertical position of the ball at this time is:

$$y_g = -\frac{g}{2} \cdot \left(\frac{L}{v_0 \cdot \cos 40°}\right)^2 + v_0 \cdot \sin 40° \cdot \frac{L}{v_0 \cdot \cos 40°}$$

To avoid bouncing, a condition $y_g > 0$ should be satisfied (mind the inequalities while ordering):

$$-\frac{g L^2}{2 v_0^2 \cos^2 40°} + L \cdot \tan 40° > 0 \rightarrow L \cdot \tan 40° > \frac{g L^2}{2 v_0^2 \cos^2 40°} \rightarrow v_0^2 > \frac{g L}{2 \tan 40° \cos^2 40°},$$

from which (knowing that $\tan\alpha \cdot \cos\alpha = \sin\alpha$, as well as $2 \cdot \sin\alpha \cdot \cos\alpha = \sin(2 \cdot \alpha)$):

$$v_0 > \sqrt{\frac{g L}{\sin 80°}} = \sqrt{\frac{9.81 \cdot 45}{\sin 80°}} = 21.17\,\text{m/s}\quad.$$

If the kick is not too high, condition $y_g < H$ is satisfied:

$$-\frac{gL^2}{2v_0^2\cos^2 40°}+L\cdot\tan 40°<H \rightarrow L\cdot\tan 40°-H<\frac{gL^2}{2v_0^2\cos^2 40°}\rightarrow v_0^2<\frac{gL}{2\left(\tan 40°-\frac{H}{L}\right)\cos^2 40°}$$

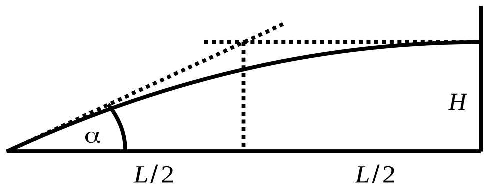
which yields

$$v_0<\sqrt{\frac{gL}{2\left(\tan 40°-\frac{H}{L}\right)\cos^2 40°}}=\sqrt{\frac{9.81\cdot 45}{2\left(\tan 40°-\frac{2.44}{45}\right)\cos^2 40°}}=21.89\,\text{m/s}$$

In summary, if the ball hits the goal directly, initial velocity of the ball must meet the double condition $\boxed{21.17\,\text{m/s}<v_0<21.89\,\text{m/s}}$.

Exercise 2

Find the initial velocity and angle required for the cannonball to hit the castle wall from $L=200\,\text{m}$ horizontally at a height of $H=14\,\text{m}$.



Solution

Deduced from the condition of impact, which component of the velocity vector is known on impact?

$$v_{...}=...\,\text{m/s}$$

How large is the acceleration in the same direction? $a_{...}=........\,\text{m/s}^2$

With this in mind, vertical component of the velocity on shot is obtained as follows:

$$v^2=v_0^2+2a\cdot s\rightarrow ....^2=v_{0y}^2...2\cdot.......\cdot.......\rightarrow v_{0y}=........\,\text{m/s}$$

Total time elapsed from shot to the impact can also be found from the vertical component:

$$s=\frac{a}{2}t^2+v_0t+s_0\rightarrow ....=\frac{\qquad}{2}t^2+........t \quad \rightarrow t_{1,2}=$$

(Because of the horizontal impact, this height is reached exactly once, that is why two roots coincide.)

What is the necessary value of horizontal velocity in order that a horizontal distance L could be covered by the cannonball (with a .... motion) in the calculated time interval?

$$L=v_x\cdot t\rightarrow ....=v_x\cdot.........\rightarrow v_x=...........\,\text{m/s}$$

From the two components of velocity on shot, the overall velocity and direction of the shot are

$$v_0=\sqrt{v_x^2+v_{0y}^2}=\sqrt{..........^2+..........^2}\rightarrow \qquad v=............$$

$$\tan\alpha=\frac{v_{0y}}{v_x}=\frac{\qquad}{\qquad}\rightarrow \qquad \alpha=............$$

Circular motion

Circular motion is a special case of plane motion when a point moves by keeping a constant distance from another given point of the plane. The position of the moving point can be given by an angle φ of rotation (called also angular displacement) with respect to a fixed initial configuration. This rotation is measured in radians and considered positive, by analogy to our coordinate system, if it is counterclockwise.

The rate of change of rotation in time is called angular velocity and denoted by ω. It is provided with the same sign convention as the rotation itself: it is positive in counterclockwise sense.

The rate of change of angular velocity in time is called angular acceleration and denoted by κ. The same sign convention is applied as in both preceding cases.

Mathematical relationships among those three parameters of motion are quite similar to those having already been noted at rectilinear motions: first time derivative of angular displacement is the angular velocity, first time derivative of angular velocity is the angular acceleration (and hence the angular acceleration is the second time derivative of the angular displacement). Based on the formal similarity of differential relationships, the following 'dictionary' can be set up:

Dictionary rectilinear – angular:

rectilinear motion	**circular motion**	*Távolságok*
x	φ	s
v	ω	v
a	κ	a_τ

All rectilinear and angular entries in this dictionary can be coordinated, making thus possible even to speak about uniform accelerating circular motion where κ = constant. In such a case, angular displacement can be obtained as a translation of the formula of the previous lecture as follows:

$$\phi(t)=\phi_0+\omega_0\cdot t+\frac{\kappa}{2}\cdot t^2$$

$$\omega(t)=\omega_0+\kappa_0\cdot t$$

Exercise: adapt the remaining two formulae as well:

$$x(t)=\frac{v_0+v}{2}\cdot t+x_0 \quad \rightarrow \quad =\frac{\quad + \quad}{2}\cdot \quad + $$

$$v^2=v_0^2+2\cdot a\cdot x \quad \rightarrow \quad {}^2= \quad {}^2+2\cdot \quad \cdot$$

Beyond the above parameters of circular motion, the motion can also be characterized by expressions of linear distance, velocity or acceleration as shown below. A curvilinear (arc) length can be calculated along the circle arc by formula $s(t)=R\cdot\phi(t)$. The velocity will always be tangential, its magnitude is $v(t)=R\cdot\omega(t)$.

Acceleration behaves in a more complicated way. When applied to rectilinear motions, it was defined as the rate of change of the velocity. Although it continues to hold, a completion of the definition is necessary in what also the direction (not only the magnitude!) of the velocity is concerned. (Recall that velocity is a vector.)

A change in magnitude only results always in a tangentia componentl of acceleration: $a_\tau(t)=R\cdot\kappa(t)$. This component, according to its positive or negative sign, can point ahead or back along the circular arc.

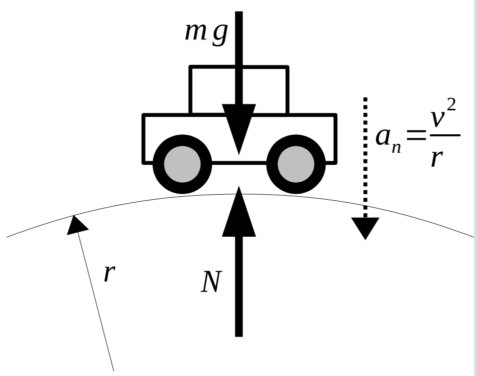
A change in the direction results always in a component of acceleration pointing towards the centre of the circle, its magnitude is $a_n(t)=\frac{v^2}{R}=\omega^2\cdot R$

The vector sum of these two perpendicular components is the resultant acceleraton of the particle.

One can easily see that there are similar differential relationships among tangential parameters to those among angular or linear parameters. First time derivative of the arc *s* completed yields velocity *v*, its further time derivative corresponds to the tangential acceleration a_τ. Our former dictionary could therefore be extended by a further column containing these three parameters; in addition, their relationships continues to be valid for *any* curvilinear motion.

Example 3

A car of mass $m=1.2\,\text{t}$ passes a hill of vertical radius $r=800\,\text{m}$ with a constant velocity of $v=70\,\text{km/h}$. Find its acceleration at the highest point of the road.



Solution

The car moves along a circle within a vertical plane. Its normal acceleration is directed towards the centre of the circle in any position; the magnitude of this component is $a_n=\frac{v^2}{r}$, the direction is always radial.

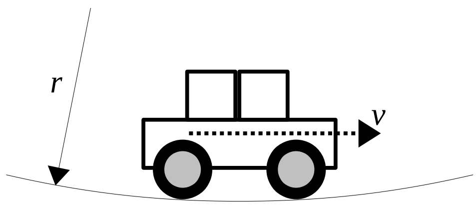
Consequently, it points vertically downwards on the top of the hill.

Velocity is $v=70/3.6=19.44\,\text{m/s}$ that yields $a_n=\frac{19.44^2}{800}\rightarrow$ $\boxed{a_n=0.4724\,\text{m/s}^2}$

Remark: It was not analysed in the problem whether or not the forces acting on the car are capable of imposing this motion: this question is already left for Kinetics. It may happen that high speed or small radius leads to a tensile force between the car and the road, required for maintaining the car on the circular path. In such a case, the car leaves the road and starts to follow a trajectory which is independent of the circular path.

Exercise 3

A car with mass of $m=1{,}0\,\text{t}$ passes a valley of vertical radius $r=800\,\text{m}$ with a constant velocity of $v=72\,\text{km/h}$. Find its acceleration at the bottom of the valley.



Solution

The car follows a circular path in a vertical plane with velocity $v=\frac{72}{3.6}=.........\,\text{m/s}$

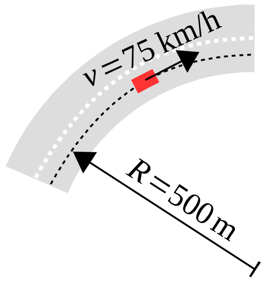
Its normal acceleration is directed towards the centre of the circle in each point of the arc. This direction in the analysed configuration is.....................

The magnitude of that normal component of acceleration is $a_n = \frac{\quad}{\quad} = \ldots\ldots\ldots \text{ m/s}^2$.

How large is the tangential acceleration? Why? $a_\tau = \ldots\ldots\ldots\ldots$

Example 4

How large is the acceleration of a vehicle following an arc of horizontal radius of $R = 500$ m with a constant velocity of $v = 75$ km/h?
Find the minimum distance required until the vehicle is fully stopped with uniform deceleration, if the resultant acceleration can never exceed 1.5 m/s^2.



Solution

If the (magnitude of) velocity is constant, there is only a normal acceleration of $a_n = \frac{v^2}{R}$, directed always towards the centre of the circle. Numeric value of the velocity converted from km/h into m/s is $v = \frac{75}{3.6} = 20.83 \text{ m/s}$, written into the formula we have $a_n = \frac{20.83^2}{500} \rightarrow$ $\boxed{a_n = 0.8678 \text{ m/s}^2}$

With an uniform acceleration, its tangential component a_τ is constant, while the normal component decreases quadratically with velocity (its initial value has just been obtained). The resultant acceleration is calculated by th Pythagorean theorem from the components as follows: $a = \sqrt{a_\tau^2 + a_n^2}$. This should be smaller than the given limit in any time instant:

$$\sqrt{a_\tau^2 + a_n^2} < 1.5 \rightarrow a_\tau^2 + a_n^2 < 1.5^2 \rightarrow a_\tau^2 < 1.5^2 - a_n^2$$

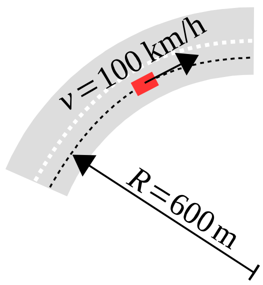
This latter inequality must also be satisfied for the (previously calculated) maximum of a_n , thus, $a_\tau < \sqrt{1.5^2 - 0.8678^2} = 1.223 \text{ m/s}^2$. Since rounding of the value has been done downwards, this rounded value can be used in further calculations.

From the calculated tangential acceleration, the distance covered from start to the complete stop can be obtained (with emphasis on that this question does not involve time) from the following formula:

$$0^2 = 20.83^2 - 2 \cdot 1.223 \cdot s \rightarrow \quad \boxed{s = 177.4 \text{ m}}$$

Exercise 4

Find the acceleration of a vehicle travelling along a circular arc of radius $R = 600$ m with a constant velocity of $v = 100$ km/h. The velocity is increased to 120 km/h in 2 seconds by uniform acceleration. Calculate the resultant acceleration both at the beginning and the end of speed-up.



Solution

How large is the constant velocity in m/s? $v = \frac{}{3.6} = ...\ \mathrm{m/s}$

In which direction does the body accelerate when moving on a horizontal circular path with a constant velocity? ....

How large is that velocity? $a_n = \frac{v^2}{R} = \frac{}{} = ...\ \mathrm{m/s}^2$

The velocity at the end of speed-up in m/s is $v = \frac{120}{3.6} = ...\ \mathrm{m/s}$

Knowing the duration of speed-up, tangential component of acceleration is obtained as:

$$v = v_0 + a \cdot t \rightarrow .... = ... + a_\tau \cdot ...... \rightarrow \quad a_\tau = ...\ \mathrm{m/s}^2$$

The resultant acceleration at the beginning of speed-up (with respect to the normal acceleration obtained for constant (magnitude of) velocity) reads:

$$a_0 = \sqrt{..........^2 + ...^2} \rightarrow \quad a_0 = ...\ \mathrm{m/s}^2$$

Resultant acceleration at the end of speed-up is similarly obtained as:

$$a_1 = \sqrt{..........^2 + \left(\frac{^2}{} \right)^2} \rightarrow \quad a_1 = ...\ \mathrm{m/s}^2$$

Motion along a prescribed path

Let geometrically admissible points (forming the trajectory) of the moving particle be given by a function $\boldsymbol{r}(s)$. Here the parameter s specifies the distance between the given position of the particle and a fixed initial one, measured along the path (s is called therefore *arc length parameter*). With this formalism it is sufficient to specify a function $s(t)$ in order to describe the entire motion in time. Based on this function $s(t)$, further parameters of motion can be obtained as shown in the following.

The vector of velocity will always remain tangent to the curved path of motion. The magnitude of velocity is the time derivative of the arc length parameter, that is, $v = \frac{\mathrm{d}s}{\mathrm{dt}}$. (A negative value of velocity refers here to a 'backwards' motion (against increasing s)).

Any motion can locally be interpreted as a motion along an osculating circle pertaining to the current point on the path; therefore, earlier considerations on circular motion still apply. The vector of acceleration has two components: *tangential* acceleration corresponds to the rate of change of magnitude of the velocity: $a_\tau = \frac{\mathrm{d}v}{\mathrm{dt}}$. The change of direction of the velocity is expressed, exactly as in circular motion, by the normal acceleration, calculated as $a_n = \frac{v^2}{\varrho}$, where v is the velocity, ϱ is the radius of curvature of the path, being always equal to the radius of the osculating circle. This component of acceleration is always perpendicular to the path and is directed towards the centre of the osculating circle.

If the motion is known through both coordinates and local parameters of the path, then the vector of acceleration obtained from either components a_x, a_y, a_z or components a_τ, a_n should be the same.

Newton's laws of motion

As a reminder, let us re-read the three laws (which are, in fact, axioms, but at the scale of parameters we use in the description of motions, they can be regarded as laws):

1. *Principle of inertial motion.* "Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by *force* impressed".

2. *Law of Dynamics.* "The rate of change of momentum of an object is proportional to the resultant force acting on the body and is in the same direction", that is, the rate of change of *velocity* (*acceleration*, $\boldsymbol{a}$) of a material particle is proportional to the force $\boldsymbol{F}$ and the factor of proportion is the (constant) mass m of the body: $\boldsymbol{F}=m\,\boldsymbol{a}$.

3. *Principle of action and reaction.* "All forces occur in pairs, and these two forces are equal in magnitude and opposite in direction." It is important to emphasize that two forces in such a pair act on two different bodies in interaction. In other words, "every action has opposite and equalent reaction".

Newton added some corollaries to these laws which were needed to replace mathematical operations that were yet unknown; they are not listed here.

We continue considering material particles only, so any force acting on a particle should have a line of action that passes through the referred point. For this reason, at any time instant of the motion, a coordinate system xyz can be set with the referred point in its origin. With the help of this, the overall action of multiple forces can simply be described by a vector sum of forces.

Here and now, only the second law of motion is applied. Two main types of problems can be distinguished: either the forces acting upon a body are known one by one and the parameters of motion should be extracted from them or it is aimed at finding some forces needed to produce a given motion. In the former case, forces on the left hand side of the equation $\boldsymbol{F}=m\boldsymbol{a}$ should be added, which yields accelerations by coordinate directions, then these accelerations together with initial values of velocity and position allow us to determine the current position. In the latter case, the procedure is reversed: first the acceleration is extractd from other parameters of motion, then unknown force components on the left hand side of $\boldsymbol{F} = m\,\boldsymbol{a}$ should be calculated. Static problems involve no motion, so they involve no acceleration either. As a consequence of this, static problems always require a calculation procedure shown in the latter case.

When this second type of problems is dealt with, i.e., forces are calculated from the acceleration, there always belong some kinematic constraint to the motion. Constraint in this context means that the motion of the body is not completely free: it is constrained in some directions. This constraint is represented by a force of unknown magnitude (those forces will later be called work compatible with, that is, capable of doing work on the constrained displacement).

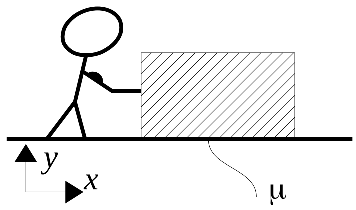
Dry (kinetic) friction

If two flat surfaces in contact undergo relative sliding, a phenomenon called *kinetic friction* can be observed, that is, the force transmitted by the contacting surfaces can be *resolved* (i.e., decomposed) into two components. The component that pushes the surfaces against each other is perpendicular to the plane of contact and is called therefore *normal force* (it is commonly denoted by N, and it should obviously be a compressive force). The component parallel to the plane of contact is called *force of friction* (denoted by F_f). The proportion of these two forces yield the *coefficient of friction*, $\mu = F_f / N$.

Although several models are known for describing friction and the coefficient of friction, here we use only simple models when μ is constant (i.e., independent both of the normal force and relative velocity of the surfaces). This modell is known as Coulomb's model for dry friction.

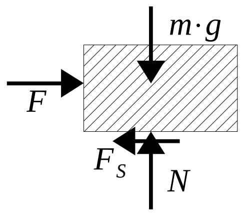
Example 5

A body with a mass of $m = 50\,\mathrm{kg}$ is accelerated by a horizontal force of $F = 200\,\mathrm{N}$.
Find the minimum length needed for accelerating the body to a velocity of $13\,\mathrm{m/s}$ if the coefficient of friction is $\mu = 0.1$?



Solution

The attached figure shows all forces acting on the body.



Because of gravity, the force $m \cdot g = 50 \cdot 9.81 = 490.\,\mathrm{N}$ is directed downwards, F points to the right, N points upwards. If the body accelerates, both the vector of acceleration and the subsequent velocity must point to the right; therefore, the friction force F_f has to point to the left. Since the body moves horizontally, both its velocity and acceleration will be horizontal, with no vertical components. Newton's second law applies here as follows:

$$\sum F_{i\rightarrow} : F - F_f = m \cdot a_x$$

$$\sum F_{i\uparrow} : N - m \cdot g = m \cdot 0$$

From the second equation: $N = m \cdot g = 490.5\,\mathrm{N}$

Using the condition (law) of friction, $F_f = \mu \cdot N = 0.1 \cdot 490.5 = 49.05\,\mathrm{N}$

Writing this into the horizontal equation: $200 - 49.05 = 50 \cdot a_x \rightarrow$ $\boxed{a_x = +3.019\,\mathrm{m/s^2}(\rightarrow)}$

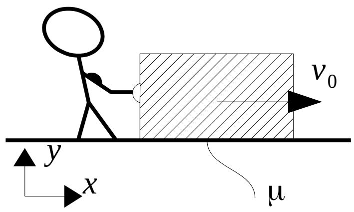
Positive sign confirms our *assumption* on the sense of acceleration (to the right). (Since the occurrence of sliding has been involved in our assumptions, a negative result would have a completely different interpretation. E.g., a friction of coefficient five times larger would result in a left-oriented acceleration, but this would reverse the velocity and therefore would contradict with the assumed sense of friction force. This contradiction is resolved by noticing that in this latter case, in fact no motion would occur: its acceleration would be zero instead, and friction force would be limited just to balance the force F attempting to move the body. If it did not start from rest, however, it would be possible a velocity to the right and acceleration to the left simultaneously.)

Let us proceed with the solution of the kinematic part of the problem. Acceleration is constant, so formulae of the rectilinear uniformly accelerating motion can be used:

$$v^2 = v_0^2 + 2a_x \cdot x \rightarrow 13^2 = 0^2 + 2 \cdot 3.019 \cdot x$$

Ennek megoldásából a gyorsításhoz szükséges út: $\boxed{x = 27.99\,\text{m}}$.

Exercise 5



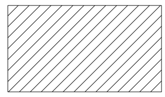
What time does it take to stop a body of $m = 50\,\text{kg}$ applying a horizontal force of $F = 200\,\text{N}$ if the initial velocity is $13\,\text{m/s}$ and the coefficient of friction is $\mu = 0.1$?

Solution

In order to solve the problem, one should calculate the acceleration and then the time required until stop.

Draw in the sketch all forces acting upon the body:

(Mind the sense of velocity when assuming the sense of the friction force)



The force of gravity can directly be obtained from the mass:

$$m \cdot g = ... \cdot ..... = ...\,\text{N}$$

The components of acceleration are the following:

$$a_y = ...$$ sense (arrow) of the horizontal component: (..........)

Write two scalar equations of Newton's second law (and solve them immediately if possible):

$$\sum F_{iy} : .... \quad .... = m \cdot 0$$

$$\sum F_{ix} : .... \quad .... = \quad m \cdot ....$$

Calculate friction force from the normal force:

$$F_f = \mu \cdot N = ... \cdot .... = ...\,\text{N}$$

Writing this into the horizontal equation we have:

$$......... \quad .... = 50 \cdot ...... \rightarrow \quad a_x = ... (...)$$

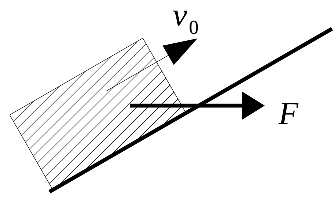
The type of motion is :... (mind the constant acceleration)

The formula containing both initial and final velocities and the acceleration reads:..........................

Numerically: $0 = 13 .... \quad .... \cdot t \rightarrow \quad t = ...\,\text{s}$

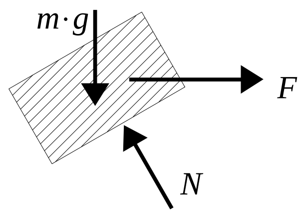
Example 6

A body of mass of $m = 15\,\text{kg}$ is pulled by a horizontal force $F = 120\,\text{N}$ against a slope of inclination $\alpha = 30\,°$.
The initial velocity at the bottom of the slope is $v_0 = 3\,\text{m/s}$.
Find the velocity at the top of the 10-m long slope.
(Friction can be neglected.)



Solution

The attached figure shows all forces acting on the body.



Because of gravity, the force $m \cdot g = 15 \cdot 9.81 = 147.2\,\text{N}$ points downwards, F points to the right, N points to the left and upwards. Since the body moves along the slope, both the vector of acceleration and the subsequent velocity will be parallel to it. Newton's second law can be written here either in horizontal and vertical directions as usual:

$$\sum F_{ix} : F - N \cdot \sin\alpha = m \cdot a \cos\alpha$$

$$\sum F_{iy} : -m \cdot g + N \cdot \cos\alpha = m \cdot a \sin\alpha$$

or in directions parallel and perpendicular to the slope:

$$\sum F_{i\searrow} : m \cdot g \cdot \cos\alpha - N + F \cdot \sin\alpha = m \cdot 0$$

$$\sum F_{i\nearrow} : -m \cdot g \cdot \sin\alpha + F \cdot \cos\alpha = m \cdot a$$

Out of the four equations, only two are needed. It is easy to see that in each of the first two equations both unknowns appear, while the last two equations contain a single unknown each. Its physical reason is that their directions are perpendicular to the normal and friction forces, respectively, making them to disappear from the respective equation. Of course, the solution obtained from either system of equations should be unique but it is easier now to adopt the latter choice: $-147.2 \cdot \sin 30° + 120 \cdot \cos 30° = 15 \cdot a \rightarrow \quad a = 2.022\,\text{m/s}^2 (\nearrow)$.

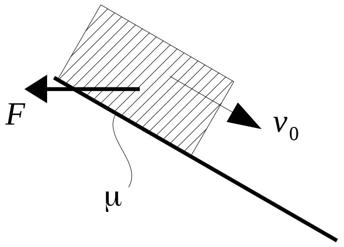
Plugging this velocity into the formula of uniform accelerating motion we get:

$$v^2 = 3^2 + 2 \cdot 2.022 \cdot 10 \rightarrow \quad \boxed{v = 7.031\,\text{m/s} (\nearrow)}$$

Remark 1: strictly speaking, the *mathematical* problem could have resulted also in a solution of -7,031 because of the square root operation, but such an answer would have been associated with a negative time, being out of scope of this problem.

Remark 2: if friction had had not been neglected in the solution, its parameters would have appeared in the equations except for the one perpendicular to the slope. For that reason, the normal component could have been calculated as we did above, then the corresponding friction force could have been used in the parallel equation. In such a case, an occasional negative root is not physically acceptable not simply because of the negative time but rather since the obtained negative velocity would imply a contradiction due to the oppositely directed friction force (and acceleration).

Exercise 6



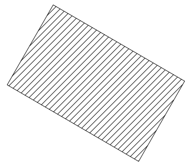
A body of mass of $m=15\,\text{kg}$ is sliding down a slope of inclination of $\alpha=30°$. Its initial velocity is $v_0=7\,\text{m/s}$ and the body is decelerated by a force $F=120\,\text{N}$. Calculate the distance required until it stops if the coefficient of friction is $\mu=0.15$.

Solution

In order to solve the problem, one should calculate the acceleration and then the distance required until stop.

Draw in the sketch all forces acting upon the body:

(Mind the sense of acceleration when assuming the sense of the friction force)



The force of gravity can directly be obtained from the mass:

$$m\cdot g=\ldots.\cdot\ldots..=\ldots\ldots\ldots\,\text{N}$$

Write four possible scalar equations based on Newton's second law:

$$\sum F_{iy}:\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots=\quad m\cdot\ldots\ldots.$$

$$\sum F_{ix}:\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots=\quad m\cdot\ldots\ldots.$$

$$\sum F_{i\nearrow}:\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots=\quad m\cdot\ldots\ldots.$$

$$\sum F_{i\searrow}:\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots=\quad m\cdot\ldots\ldots.$$

Solve the equation with one unknown only:

$$N=\ldots\ldots\ldots\ldots\,\text{N}$$

Calculate friction force from the normal component:

$$F_f=\mu\cdot N=\ldots..\cdot\ldots\ldots=\ldots\ldots.\,\text{N}$$

There is only one unknown left, calculate this by solving any equation:

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \rightarrow \quad a_x=\ldots\ldots\ldots\ldots.(\ldots)$$

The type of motion is :... (mind the constant acceleration)

The formula with both initial and final velocities, distance and acceleration reads:.........................

Numerically: $0^2=7^2-2\cdot\ldots\ldots\cdot s \quad\rightarrow\quad s=\ldots\ldots\ldots\ldots\,\text{m}$

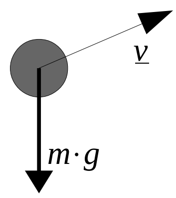
Notes:

Application of Newton's laws of motion in planar problems

This lecture will be dedicated to the calculation of resultant forces acting on a body, completed by the calculation both of the accelerations due to those forces and/or of forces preventing the body from acceleration. In contrast to the past lecture, the material particle will be constrained to a plane instead of a line henceforth.

Example 1

Calculate the forces acting on a thrown body regarded as a material particle if the air resistance is neglected.



Solution

By the neglection of air resistance, only the gravitational force acts on the body (downwards) with a magnitude of $m \cdot g$.

This force is always directed downwards vertically and its magnitude depends on mass *m* of the body as well as the gravitational acceleration.

This latter parameter in general depends on both position and time; in this course it is uniformly taken as $g = 9.81\,\text{m/s}^2$.

Exercise 1

Determine the resultant of forces acting on a projectile shot by a cannon. The cannonball is a solid sphere made of steel whose diameter is 25 cm and density is $\varrho = 7850\ \text{kg/m}^3$, air resistance can be neglected.

Solution

How many forces are taken into accound if air resistance is neglected?................................

The magnitude of gravitational force is $m \cdot g$.

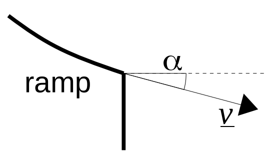
How large is the volume of the sphere? $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi \ldots\ldots^3 = \ldots\ldots\ldots\ldots \text{cm}^3$

The mass of the cannonball is $m = \varrho \cdot V = 7.85 \cdot \ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

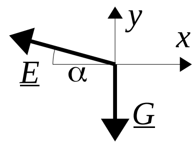
The gravitational force acting upon the cannonball is $m \cdot g = \ldots\ldots\ldots\ldots \cdot 9.81 = \ldots\ldots\ldots\ldots$

Example 2

A ski jumper sinks at a given instant with a velocity of $v = 90\,\text{km/h}$ under an angle $\alpha = 7°$ with respect to the horizontal direction. Air resistance is represented by a force opposed to the velocity and possessing a magnitude of $c v^2 \varrho$, where $c = 0.3\,\text{m}^2$ is a factor dependent on shape and extension and $\varrho = 1.3\,\text{kg/m}^3$ is the air density. Calculate the resultant force acting on the ski jumper, as well as the acceleration if the mass of the jumper is $m = 75\,\text{kg}$.



Solution

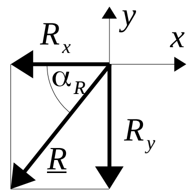


In the position sketched in the above figure, two forces act on the jumper, see anothe figure to the right. The (vertical) force of gravity has a magnitude of $G=m\cdot g=75\cdot 9.81=735.8\,\mathrm{N}(\downarrow)$.

The force due to air resistance:

$$E=0.3\cdot\left(\frac{90}{3.6}\right)^2\cdot 1.3=243.8\,\mathrm{N}(\nwarrow)$$

The resultant force acting on the jumper is



$$\underline{R}=\underline{G}+\underline{E}=\begin{bmatrix}0\\-735.8\end{bmatrix}+\begin{bmatrix}-243.8\cdot\cos 7°\\243.8\cdot\sin 7°\end{bmatrix}\rightarrow\quad\boxed{\underline{R}=\begin{bmatrix}-242.0\\-706.1\end{bmatrix}\mathrm{N}}\ ,$$

its magnitude, $|\underline{R}|=\sqrt{242.0^2+706.1^2}\rightarrow\ \boxed{R=746.4\,\mathrm{N}(\swarrow)}$,

its angle to the horizontal is $\tan\alpha_R=\dfrac{706.1}{242.0}\rightarrow\ \boxed{\alpha_R=71.08°}$.

The acceleration can be calculcted from the resultant force. With reference to the vector equation $\underline{R}=m\cdot\underline{a}$, the vector of acceleration points to the left and downwards and encloses an angle of 71.08° with the horizontal and its magnitude is

$$746.4=75\cdot a\rightarrow\ \boxed{a=9.952\,\mathrm{m/s^2}}\ .$$

Two components of acceleration could also be obtained directly, using two resolution equations (for convenience, in directions *x* and *y*). Positive orientation can be set arbitrarily in each equation but it is common to choose the positive direction along the positive sense of acceleration and to write the signs of other terms accordingly. An advantage of this choice in dynamical problems is that one cannot *miss* the sign of the right-hand-side term ***ma*** since it is always positive. In the present case, let the acceleration be assumed to point downwards vertically and to the right horizontally, then write two resolution equations in this sense as follows:

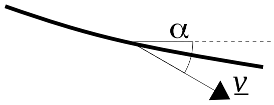
$$\sum F_{i\downarrow}: 735.8-243.8\cdot\sin 7°=75\cdot a_y\ \rightarrow\quad\boxed{a_y=9.415\,\mathrm{m/s^2}(\downarrow)}$$

$$\sum F_{i\rightarrow}: 0-243.8\cdot\cos 7°=75\cdot a_x\ \rightarrow\quad\boxed{a_x=-3.226\,\mathrm{m/s^2}(\leftarrow)}$$

Positive or negative signs of solutions refer always to that effective direction (sense) of the calculated variable agrees or contradicts, respectively, with their assumed sense. Thus, the resultant acceleration points downwards and to the left: it is explicitly emphasized by the arrow in brackets behind the numeric result..

Starting with these components of acceleration and using the Pythagorean theorem as well as the arctan function, the same results are obtained.

Exercise 2



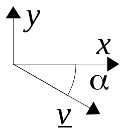
The velocity of a ski jumper is inclined at an angle $\alpha=22°$ to the horizontal and its magnitude is $v=84$ km/h just before landing. The force due to air resistance is $c\,v^2\varrho$, opposed to the velocity, where $c=0.25\,\mathrm{m^2}$ is a factor dependent on shape and extension, $\varrho=1.3\,\mathrm{kg/m^3}$ is the density of air. The jumper weighs $m=75$ kg. Find the resultant force and acceleration acting on the ski jumper.

Solution

The velocity vector of the jumper is shown in the figure to the right.

Draw into the figure the following forces:

- vertical force $\underline{G}$ due to weight,
- the force $\underline{E}$ of air resistance.



The force due to weight is $G=m\cdot g=$

Velocity in m/s: $v=\dfrac{84}{3.6}=$

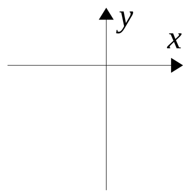
The air resistance is $E=0.25\cdot 23.33^2\cdot 1.3=$

The jumper is acted upon by the sum of two previous forces: $\underline{R}=\underline{G}+\underline{E}$. Draw also the two assumed components of resultant force R into the figure, then write and solve horizontal and vertical resolution equations:

$$\sum F_{ix}:\ldots R_x=\ldots\ldots+\ldots\ldots\cdot\ldots\ldots\ldots \quad\rightarrow R_x=$$

$$\sum F_{iy}:\ldots R_y=\ldots\ldots+\ldots\ldots\cdot\ldots\ldots\ldots \quad\rightarrow R_y=$$

The magnitude of the resultant is $R=\sqrt{R_x^2+R_y^2}=\sqrt{\ldots\ldots^2+\ldots\ldots^2}=$



and it is inclination to the horizontal is $\tan\alpha_R=\text{——}\rightarrow\alpha_R=$

In order that further calculations should not be influenced by (occasional) earlier miscalculations, acceleration is obtained from two resolutions of the vector equation $\underline{G}+\underline{E}=m\cdot\underline{a}$. Let the components of $\underline{a}$ be assumed to point left and downwards. In order to prevent the sign of acceleration from being confused on the right hand side, let these assumed directions be considered positive in the resolution equations:

$$\sum F_{i\downarrow}:+\ldots\ldots\ldots-\ldots\ldots\ldots\ldots\cdot\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\cdot a_y$$

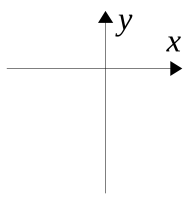
$$\sum F_{i\leftarrow}:+\ldots\ldots\ldots+\ldots\ldots\ldots\ldots\cdot\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\cdot a_x$$

Solving the above equations one by one (mark also the effective direction in brackets):

$$a_y=\ldots\ldots\ldots\ldots\ldots\ldots\ \mathrm{m/s^2}(\quad) \qquad a_x=\ldots\ldots\ldots\ldots\ldots\ldots\ \mathrm{m/s^2}(\quad)$$

The magnitude of acceleration is

$$a=\sqrt{\ldots\ldots^2+\ldots\ldots\ldots^2}=$$



direction: $\mathrm{tg}\,\alpha_a=\text{——}\rightarrow\alpha_a=$

Safety check:

Total force generating acceleration is $m\cdot a=\ldots\ldots\ldots\ldots\cdot\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots$.

Do this force and the resultant coincide? If not, why and in what amount?

Do directions of the resultant force and acceleration coincide? If not, why and in what amount?

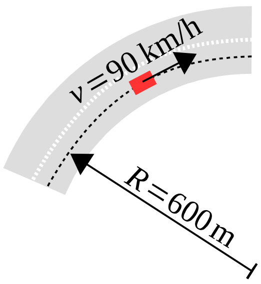
Dry (static) friction

In some cases, two bodies in contact do not have relative motion in their contact point(s): they do not slide with respect to each other. This phenomenon is called *static friction* (in contrast to *kinetic friction* discussed in the previous lecture), which will be described here by the simplest model of dry friction introduced by Coulomb. Its main assumption is that two surfaces do not move on each other unti the proportion of fricion force and normal force component does not reach the coefficient of friction; it can formally be written by inequality $|F_f|/N\le\mu$. (Or, in other terms, if the friction force does not reach the product of normal force and the coefficient of friction, $|F_S|\le\mu N$.) Static friction force is directed always against the motion it prevents, but otherwise it can be considered as a free parameter until it reaches its limit. The minimum value of coefficient of friction in order to prevent the surfaces from sliding can therefore be found from the above inequalitiy.

Road vehicles proceed by rolling wheels. Its dynamics are not discussed here but it is obvious that friction force between wheels and the road must remain static to ensure the driver's control over the vehicle instead of a slipping governed just by the inertial mass and kinetic friction force (this latter way of control requires considerably more space than is actually available on roads...).

Example 3

A vehicle of $m=1.5$ t, modelled as a particle, proceeds with a velocity of $v=90$ km/h along an arc with radius $R=600$ m.
Calculate the forces acting on the vehicle.
Is a static friction coefficient $\mu=0.2$ sufficient to prevent the vehicle from slipping?



Solution

The result can be obtained in three steps.

In the first round, components of the net acceleration of the body should be calculated; in the second, unknown forces should be determined from accelerations using Newton's second law. Finally, the last question can be answered considering the law of friction.

Accelerations

The vehicle follows a circular path in a horizontal plane, so it has neither vertical nor tangential acceleration. The only nonzero component is the normal acceleration directed towards the centre of the circle:

$$a_n=\frac{v^2}{R}=\frac{(90/3.6)^2}{600}=\frac{25^2}{600}=1.042\,\mathrm{m/s^2}$$

Forces

The resultant force acting on the body can be resolved into three components. Gravity acts vertically downwards with a magnitude of $m\cdot g=1.5\cdot 9.81\rightarrow$ $\boxed{m\cdot g=14.72\,\mathrm{kN}}$.

Let the vertical component of the force exerted by the road on the vehicle be denoted by *N*, this force is directed upwards. A vertical resolution equation of Newton's second law reads: $m \cdot g - N = m \cdot 0$, where forces directed downwards are taken as positive; zero value on the right hand side is due to the above-referred zero vertical acceleration. Solution to this problem is

$N = m \cdot g \rightarrow$ $\boxed{N = 14.72\,\text{kN}}$ (The positive sign confirms that *N* points upwards indeed.)

The horizontal component of the force exerted by the road on the vehicle comes from friction and will therefore be denoted by F_f . Since this is the only horizontal force in the problem, its direction must coincide with that of the acceleration, pointing towards the circle centre. Writing a resolution equation of Newton's second law in this drection we have

$F_f = m \cdot a_n = 1.5 \cdot 1.042 \rightarrow$ $\boxed{F_f = 1.563\,\text{kN}}$

It is emphasized that forces exerted on the vehicle by the road are the consequence of an interaction between the vehicle and the road. But, according to Newton's third law, the road itself is acted upon the negatives of *N* and F_f; that is, a force that is directed downwards with magnitudel 14.72kN and another but horizontal one with magnitude 1.563kN and with a radial outwards direction.

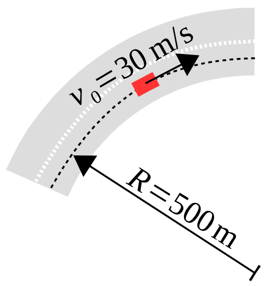
All the above calculation is based on the assumption of static friction. The given normal force component, however, implies an upper limit for the friction force component : $F_f^{max} = \mu \cdot N = 0.2 \cdot 14.72 = 2.944\,\text{kN}$. Since it wa s shown that the analysed motion can be ensured even by a smaller force (1.563<2.944), the answer for the last question sounds 'yes, it is'.

This latter question could have also been decided by the proportion F_f / N as well:

$\frac{1.563}{14.72} = 0.1062$. This value must not exceed the static coefficient of friction. This condition is satisfied (0.1062<0.2), hence one gets to the same conclusion as before.

Exercise 3

A vehicle of $m = 1.8$ t, modelled as a particle, proceeds with a velocity of $v = 30$ m/s along an arc with radius $R = 500$ m, then stops in $t = 8$ s with uniform deceleration. Calculate the forces acting on the vehicle when it starts braking. Is a static friction coefficient $\mu = 0.4$ sufficient to stop the vehicle?



Solution

In the solution process, accelerations and related forces should be determined, then the condition of friction checked.

Accelerations

Which is the plane the vehicle moves in? ....

How large is the acceleration in a direction perpendicular to that plane? $a_{...} = ...\ \text{m/s}^2$

How large is the tangential acceleration? $... = ... + a_\tau \cdot ... \rightarrow a_\tau = ...\ \text{m/s}^2$

How large is the normal component of acceleration when the braking starts?

$$a_n = \frac{....^2}{....} = ...\ \text{m/s}^2$$

In that time instant, the net acceleration of the vehicle is

$$a=\sqrt{a_n^2+a_\tau^2}=\sqrt{.......^2+.......^2}=.........\ \text{m/s}^2$$

Forces

Which of the forces acting on the vehicle has a vertical component? ....

The weight of the vehicle is $m\cdot g=..................$

The vertical resolution according to Newton's second law:

$$...........-............=m\cdot......\rightarrow N=.............$$

The only one horizontal force exerted on the body is the force of friction.

The resolution equation along the direction of acceleration based on Newton's second law:

$$F_f=......\cdot.....\rightarrow F_f=..........\ \text{kN}$$

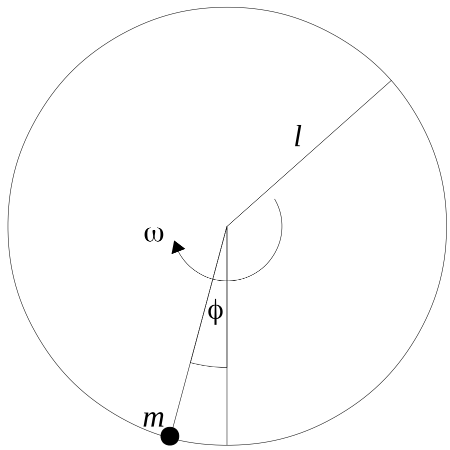
What is the maximum for friction force with the given value of coefficient of friction?

$$F_f^{max}=\mu\cdot N=.............\cdot..........=...........................\text{kN}$$

Compared to the coefficient of friction required for the prescribed motion, does that coefficient of friction suffice?...........................

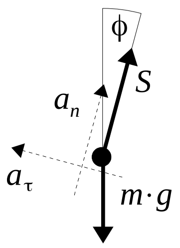
Example 4

A body of mass $M=10\,\text{kg}$ is whirled on a chord of length $l=1.3\,\text{m}$ in a vertical plane. At a given instant the chord encloses an angle of $\varphi=10°$ with the vertical direction and moves with $v=15\,\text{m/s}$ just having left the bottom point.
Calculate the acceleration of the body and the force in the cable in that position. Find all missing parameters of the circular motion.



Solution

In the position specified by the question, two forces: weight and cable force are acting on the body as shown in the attached figure. The former one is directed downwards vertically with magnitude $G=M\cdot g=98.1\,\text{N}$. The cable force always follow the direction of the cable and points now therefore towards the circle centre with an unknown magnitude *S*. These to forces together generate accelerations required for the circular motion: normal and tangential components which are shown in dashed lines in the figure.



The normal acceleration is obtained as follows:

$$a_n=\frac{v^2}{R}=\frac{15^2}{1.3}=173.1\,\mathrm{m/s^2} \quad .$$

The tangential acceleration is unknown, its direction is assumed according to the figure.

Newton's second law implies the existence of two independent equations. These could be written along the common horizontal and vertical resolutions:

$$\begin{aligned}&\sum F_{iy}: S\cdot\cos 10° - 98.1 = 10\cdot(173.1\cdot\cos 10° + a_\tau\cdot\sin 10°),\\ &\sum F_{ix}: S\cdot\sin 10° + 0 = 10\cdot(173.1\cdot\sin 10° - a_\tau\cdot\cos 10°),\end{aligned} \quad ,$$

but a rotated coordinate system can also be used where one equation is written in radial (positive if points to the right and up), the other one in tangential (positive if points left and up) directions:

$$\sum F_{in}: S - 98.1\cdot\cos 10° = 10\cdot 173.1 \quad ,$$

$$\sum F_{i\tau}: 0 - 98.1\cdot\sin 10° = 10\cdot a_\tau \quad .$$

Any two out of the four equations written above yields the same solution; the simplest choice is to work with the last two, since they have a single unknown each. From them we have

$$\boxed{S=1828\,\mathrm{N}} \quad , \quad \boxed{a_\tau=-1.703\,\mathrm{m/s^2}(\searrow)}$$

Missing parameters of the circular motions are the angular velocity and the angular acceleration. The first one can be calculated from the velocity and the radius of the circle, deciding on its direction by inspection:

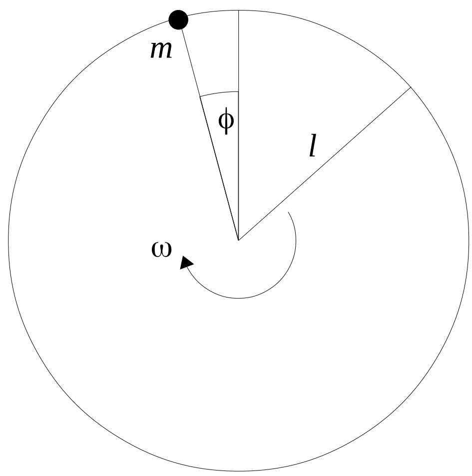
$$\omega=v/l=11.54\,\mathrm{rad/s}\ (\curvearrowright) \quad .$$

Angular acceleration can be obtained from the tangential acceleration and the radius again. The direction of tangential acceleration implies a counterclockwise angular acceleration:

$$\kappa=a_\tau/l=1.31\,\mathrm{rad/s^2}(\curvearrowleft)$$

Exercise 4

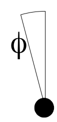
A body of mass $M=10\,\mathrm{kg}$ is whirled on a chord of length $l=1.3\,\mathrm{m}$ in a vertical plane. At a given instant the chord encloses an angle of $\varphi=10°$ with the vertical direction and moves with $\omega=6\,\mathrm{rad/s}$ just before reaching the top point.
Calculate the acceleration of the body and the force in the cable in that position. Find all missing parameters of the circular motion.



Solution

Firstly, let a sketch of the whirling body be drawn; display all forces acting on the body which are as follows:

- weight $\underline{G}$,
- cable force $\underline{S}$,



as well as two components of the acceleration:

- the normal component (a_n),
- the tangential component(a_τ).

From these four data it is known or can directly be calculated:

- the weight: $G=m\cdot g=..........\cdot.........=..........$ N
- the normal acceleration: $a_n=\omega^2\cdot l=.........^2\cdot..........=...........$ m/s^2

Out of the two further unknowns, cable force is asked by the problem, and the tangential acceleration is directly related to angular acceleration: $a_\tau=........\cdot........$, that is, by getting the former one, the latter one can also be determined.

In orde to calculate the two unknowns, two scalar projections (resolutions) of the vector equation $\underline{R}=m\cdot\underline{a}$ should be set up and solved. Those can be written along the horizontal and vertical direction, as well as in radial and tangential directions. In the present case, all four equations are written, but since it is always practical to deal with equations in only one variable, it should always be considered for each equation which out of all unknowns (i.e., S and a_τ) *will appear* in the equation and which ones *will not*.

Horizontal resolution; unknowns excluded:....................., unknowns included:.....................

$$\sum F_{i\rightarrow}:..............................\ ....=.........\cdot(..............\quad............)$$

Vertical resolution; unknowns excluded:....................., unknowns included:.....................

$$\sum F_{i\downarrow}:.............................\ ....=.........\cdot(..............\quad............)$$

Radial resolution; unknowns excluded:....................., unknowns included:.....................

$$\sum F_{i\searrow}:.........................\ ....=.........\cdot.................$$

Tangential resolution; unknowns excluded:....................., unknowns included:.....................

$$\sum F_{i\nearrow}:.........................\ ....=.........\cdot.................$$

Solve the two simplest equations:

$$S=.................\ \mathrm{N}$$

$$a_\tau=..........\ \mathrm{m/s^2}(...)\rightarrow\qquad\kappa=..............\ \mathrm{rad/s^2}(...)$$

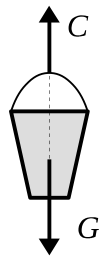
Remark: A negative sign in the cable force would mean a conpressed cable. This cannot happen in reality, that is why either the body should be whirled quicker or a rod should be used instead of a cable.

Angular velocity at the specified instant: $\omega=\frac{v}{l}=\frac{}{}=.........\ \mathrm{rad/s}(...)$

Angular acceleration at the specified instant: $\kappa=\frac{a_{...}}{l}=\frac{}{}=.........\ \mathrm{rad/s^2}(...)$

Balancing

Recall the comments of the past lecture on two kinds of problems with forces and accelerations: either accelerations caused by given forces are sought or, conversely, forces generating a prescribed acceleration are looked for. A special type of the latter problem is when the prescribed acceleration is zero (that, is, the right hand side of the equation of Newton's second law is zero). If the initial velocity is also set to zero, then the body remains in rest and will be in equilibrium. Consequently, problems of this form are called balancing problems.

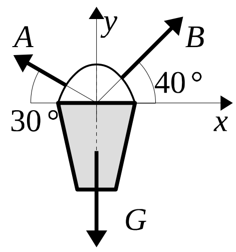


The simplest problem of balancing is when the equilibrium is maintained by a single force. In such a case only one force should be added to the resultant of all *active forces* exerted on the body, such that the resultant of the previous (partial) resultant and the balancing force should be zero. In other terms it means that the balancing force must be the negative of the resultant of all other forces.

For example, if one wants to balance a bucket of a weight of $G=0.2\,\text{kN}$ by a single force *C*, then the partial resultant of active forces will be the vertical force *G* pointing downwards. In order that it can be balanced, one needs a force of dimension of *G* having the same line of action but being directed upwards; that is, $\boxed{C=0.2\,\text{kN}}$ is obtained.

Example 5

A bucket of weight $G=0.2\,\text{kN}$ is held in equilibrium by two cables as shown in the figure. Find the forces needed for balancing.



Solution

In the case of equilibrium, the mass is always multiplied by zero on the right hand side of the equation of Newton's second law. Equilibrium equations can be written in horizontal and vertical directions:

$$\sum F_{ix}: -A\cdot\cos 30° + B\cdot\cos 40° + 0 = 0$$

$$\sum F_{iy}: A\cdot\sin 30° + B\cdot\sin 40° - 0.2 = 0$$

B (or *A*) can be expressed from the first equation and can then be plugged into the second one:

$$\sum F_{iy}: A\cdot\sin 30° + A\frac{\cos 30°}{\cos 40°}\cdot\sin 40° - 0.2 = 0 \rightarrow \boxed{A=0.1630\,\text{kN}\,(\nwarrow)}$$

$$\sum F_{iy}: B\frac{\cos 40°}{\cos 30°}\cdot\sin 30° + B\cdot\sin 40° - 0.2 = 0 \rightarrow \boxed{B=0.1843\,\text{kN}\,(\nearrow)}$$

Another option for writing and solving such a system of equations is when one looks for equations in only one variable. In our example it means that an equation that is set up to be solved for *A* should not contain any component of *B*. If the system of forces have a common point of intersection (such forces are termed *concurrent* forces), it is sufficient to write and solve resolution equations only. The equation from which *B* is completely excluded is the resolution perpendicular to *B*.; that is written along a direction at an angle of 40° to the vertical (*y*) axis. It is also needed to attach a positive sense to the resolution equation; let the direction to the right and downwards be positive:

$$\sum F_{i\searrow}: -A\cdot\cos 20° + 0.2\cdot\cos 40° = 0 \rightarrow \quad A=0.1630\,\text{N}\,(\nwarrow)$$

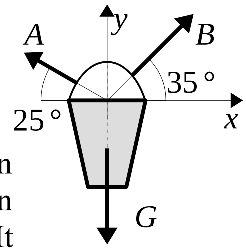
In order to calculate force *B*, a resolution not including *A*; therefore, perpendicular to *A*, should be set up and solved. This resolution is written along an axis at an angle of 30° to the vertical axis; among the two possibilities, let the direction to the right and upwards be chosen as positive:

$$\sum F_{i\nearrow}: B\cdot\cos 20° - 0.2\cdot\cos 30° = 0 \rightarrow \quad B=0.1843\,\text{N}\,(\nearrow)$$

New and old results are, of course, identical; assembling the equations required a bit more thinking but their solution was considerably faster in the latter way.

Exercise 5

A bucket of weight $G=0.2\,\mathrm{kN}$ is held in equilibrium by two cables as shown in the figure. Find the forces needed for balancing.



Solution

Because of the balancing problem, Newton's second law should be written in a form $\underline{R}=\underline{0}$, which means that two resolutions of the vector equation $\underline{G}+\underline{A}+\underline{B}=\underline{0}$ should be written and solved for two unknown magnitudes. It can be done by setting up a horizontal and a vertical equation (mind the signs and components), that can then be solved:

$$\sum F_{i,x}: 0-\ldots\ldots\ldots\cdot\ldots\ldots\ldots+\ldots\ldots\ldots\ldots\cdot\ldots\ldots\ldots=0$$

$$\sum F_{i,y}: \ldots 0.2 \;\; .. \;\; \ldots\ldots\ldots\cdot\ldots\ldots\ldots \;\; .. \;\; \ldots\ldots\ldots\ldots\cdot\ldots\ldots\ldots=0$$

As a first tep to the solution, let some unknown be expressed from the first equation in terms of another: $\ldots\ldots=\ldots\ldots\ldots\cdot\frac{\qquad}{\qquad}=\ldots\ldots\ldots\cdot\ldots$, and let it be plugged into the second equationl: $-0.2+\ldots\ldots\ldots\cdot\ldots\ldots\ldots+\ldots\ldots\ldots\ldots\cdot\ldots\ldots\ldots=0$.

Its solution is $\ldots=\ldots\ldots\ldots\,\mathrm{kN}$, which can be written back into the formula expressing the relationship between unknowns, hence $\ldots=\ldots\ldots\ldots\,\mathrm{kN}$.

This problem could also be solved in an alternative way just by the use of two equations in one variable each. In order to get force A, an equationthat does not include ..... should be chosen.

This direction encloses an angle of ..... with the vertical axis. Write the resolution along this line where the direction to the right and downwards is taken positive. Solve now the equation:

$$\sum F_{i\searrow}: +0.2\cdot\ldots\ldots-A\cdot\ldots\ldots\ldots=0 \rightarrow A=\ldots\ldots\ldots\ldots\ldots$$

Calculation of force B is done independently. The equation not including B is looked for; its direction encloses an angle of ..... with the vertical axis. The positive sense is fixed in the direction to the right and upwards. Solve this equation as well:

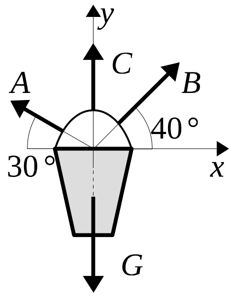
$$\sum F_{i\nearrow}: -0.2\cdot\ldots\ldots+B\cdot\ldots\ldots\ldots=0 \rightarrow B=\ldots\ldots\ldots\ldots\ldots$$

(Do not forget to specify units and effective senses (arrows) deduced from plus or minus signs at the final results.)

If one wants to balance a body in 2D by three concurrent forces of known line of action, then the solution to that problem is not unique. This feature is called *statical indeterminacy* or *hyperstatic* behaviour and will be discussed later.

Example 6

A bucket of weight $G = 0.2\,\text{kN}$ is held in equilibrium by three cables as shown in the figure. Show some possible solutions using the results of preceding examples. Determine balancing forces A and C in function of force B.



Solution

Two solutions to the problem have already been given earlier.

The figure preceding Example 5 can be completed by *zero forces* (i.e., forces of zero magnitude), A and B without the equilibrium getting modified, that is, force triplets $\boxed{A = B = 0\,\text{kN},\ C = 0.2\,\text{kN}}$ as well as $\boxed{A = 0.1630\,\text{kN}, B = 0.1843\,\text{kN}, C = 0\,\text{kN}}$ are both possible solutions to the problem. (They can easily be checked if plugged into equilibrium equations to be written just below.)

Generally speaking, equilibrium equations (resolution equations) can be written in horizontal and vertical directions:

$$\sum F_{ix}: -A\cdot\cos 30° + B\cdot\cos 40° + C\cdot\cos 90° + 0 = 0$$

$$\sum F_{iy}: A\cdot\sin 30° + B\cdot\sin 40° + C - 0.2 = 0$$

'Luckily', coefficient of C in the first equation is zero, so A can be expressed from it as follows:

$$A(B) = B\cdot\frac{\cos 40°}{\cos 30°} \rightarrow \quad \boxed{A(B) = 0.8846\cdot B}\ ,$$

then it can further be used in the second equation:

$$0.8846\cdot B\cdot\sin 30 + B\cdot\sin 40° + C - 0.2 = 0 \rightarrow \quad \boxed{C(B) = 0.2 - 1.085\cdot B}$$

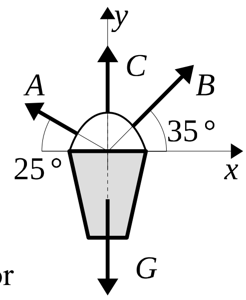
In order that the equations can be solved more easily, it can come quite in handy towrite an equation not including A when looking for C. From the two possibilities, let an axis at an angle of 30° to the vertical with a positive sense to the right and upwards be chosen:

$$\sum F_{i\nearrow}: B\cdot\cos 20° + C\cdot\cos 30° - 0.2\cdot\cos 30° = 0 \rightarrow \quad C(B) = 0.2 - 1.085\cdot B$$

The results are, of course, identical.

Exercise 6

A bucket of weight $G = 0.2\,\text{kN}$ is held in equilibrium by three cables as shown in the figure. Show some possible solutions using the results of preceding examples. Determine balancing forces B and C in function of force A.



Solution

Equilibrium of four forces should be ensured, that is, the vector equation $\underline{G}+\underline{A}+\underline{B}+\underline{C}=\underline{0}$ should be satisfied.

Completing the forces in Exercise 5 by a force C of magnitude zero (zero force), the equilibrium is not modified, thus, one possible solution is: $A = ...\,\text{N}, B = ...\,\text{N}, C = 0\,\text{N}$.

With similar arguments, if force G was balanced by a single force C, then the other two forces will be zero each: $A = ...\,\text{N}, B = ...\,\text{N}, C = 200\,\text{N}$.

In addition to these two cases shown, there are still infinitely many possibilities; they will be expressed in a form of a function. Let the independent variable of that function be the force A, hence only forces B and C are considered unknowns. For the calculation of force B, let the resolution not including C be written down. This is obviously the resolution perpendicular to C:

$$\sum F_{ix}: \ldots\ldots\ldots \;\; ..A\cdot\ldots\ldots \;\; .. \;\; B\cdot\ldots\ldots\ldots=0 \;,$$

whose solution reads $B(A)=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

For the calculation of force C, a resolution not including B (that is, a resolution perpendicular to B) should be applied; this direction is at an angle of ….. from the vertical axis. Let the direction to the right and down be chosen positive, therefore

$$\sum F_{i\searrow}: ..0.2\cdot\ldots\ldots \;\; .. \;\; A\cdot\ldots\ldots \;\; .. \;\; C\cdot\ldots\ldots=0 \;,$$

whose solution is again: $C(A)=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$.

This solution clearly demonstrates the advantage of writing equations in only one variable. Mind the difference the use of horizontal and vertical equations instead would have been made:

A horizontal resolution would have the same appearance as before, hence it is not repeated here. The vertical resolution with a positive direction taken upwards reads:

$$\sum F_{iy}: -0.2+A\cdot\ldots\ldots.+B\cdot\ldots\ldots\ldots.+C\cdot\ldots\ldots\ldots.=0$$

Although thee is no need here for solving a parametric (!) system of two equations in two variables because of the special direction of force C (i.e., B can be expressed directly from the horizontal resolution equation, as has been precisely done above). This expression of B should be plugged into the vertical resolution to have

$$\sum F_{iy}: -0.2+A\cdot\ldots\ldots.+(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)\cdot\ldots\ldots\ldots.+C\cdot\ldots\ldots\ldots.=0$$

Finally, it should be solved for C: $C(A)=\ldots\ldots\ldots\ldots\ldots\ldots$

Observations and remarks:

The sign of the result in all preceding examples gave a unique indication on whether or not the sense (direction) of the obtained variable obeys the previous assumption. This confirmation or refutation of the assumption was explicitly marked by bracketed arrows after numeric answers in all earlier occurrences. Now the sign and thus the effective sense of the result depends on the (signed) magnitude of A, that is why it is not given here.

The three reactions maintain equilibrium with G, or in other words, their sum is the negative of G: a vertical 0.2-kN force directed upwards vertically. This property was found to apply for two (called trivial) solutions as well (i.e., when A and B were zero and C was equal to $-G$, or when C was zero and A, B were not). Any linear combinations of these two equilibrium force systems yield also an equilibrium force system, and original value of G (the only active force) is exactly restored when the sum of coefficients in the combination is 1. For example, the sum of 0.3 times the first trivial solution and 0.7 times the second trivial solution also satisfies the condition of equilibrium. Let us check it...

Introduction

In some problems of kinetics studied so far, some results obtained in the course of solution are disregarded by the problem statement: some initial and final parameters are asked directly. In such cases, the use of so-called theorems of change come in handy. It can be stated as a basic principle that a theorem of change of *something* declares the change of value of something (i.e., the difference between latter and former values) to be equal to *another* quantity. (Hopefully none asserts that a theorem of change of pudding would state that the change of pudding equals the difference between latter and former values of the pudding.)

Theorem of change of linear momentum

Definition: The vector obtained as a product of mass and the velocity of a material particle is called linear momentum of the particle.

Definition: The impulse (vector) of a force on a material particle is the time integral of that force (vector). (The integral of a vector is a vector composed of integrals of each of the scalar components of the vector. If the force is constant, this integral reduces to the product of the force and the length of time interval.)

Theorem: The change of linear *momentum* of a particle caused by a force (vector) in a time interval equals the impulse of the force. It is also known as the *impulse–momentum theorem*.

Since in our problems the mass does not change, the above theorem can be reduced to the form: $m\boldsymbol{v}_2 - m\boldsymbol{v}_1 = \int_{t_1}^{t_2} \boldsymbol{R}\,\mathrm{d}t$, or, with a constant force, $m\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_1) = \boldsymbol{R}(t_2 - t_1)$.

Usage

As clearly seen from the formula, there is neither acceleration nor distance (or position coordinate) in it; consequently, the theorem can be used in any problems where the referred quantities do not appear either among given or requested parameters.

Example 1

A material particle of mass of $20\,\mathrm{kg}$ follows a rectilinear path with a velocity of $v_0 = 12\,\mathrm{m/s}$. Find the force magnitude required to double that velocity in 9 seconds.

Solution

Since it has been doubled, the final velocity (i.e., in the end of the time interval) is known:

$$v = 2\cdot v_0 = 24\,\mathrm{m/s}$$

The problem statement does not refer either to the acceleration or to the distance covered, hence the equation in a single unknown variable can be written using the theorem of change of linear momentum. However, only a scalar equation along the direction of motion is considered:

$$m(v - v_0) = F\cdot t \rightarrow 20\cdot(24 - 12) = F\cdot 9$$

(In this scalar equation, direction of both velocities and forces are reflected by their signs: the positive sense is set to the sense of motion; force F is assume to point along the velocity.)

The equation solves to $\boxed{F = +26.67\,\mathrm{N}}$.

Remark: The problem could also be solved by determining first the acceleration from the change of velocity, then by finding the force from Newton's second law. Altogether, the method presented here is far more elegant.

Exercise 1

A body of mass of $2.8\,\mathrm{t}$ is accelerated from rest for 5 minutes by a constant force $F=7.5\,\mathrm{kN}$. Find the velocity reached by the body at the end of the period.

Solution

Collect all parameters given in the problem specification:

Which quantities related to the motion are disregarded by the problem?

The theorem of change of linear momentum reads

$$m\cdot(v-v_0)=F\cdot t \rightarrow \quad \cdot(v- \quad)= \quad \cdot$$

It solves to

$$v=$$

Example 2

A material particle of mass of 25 kg is thrown in an oblique direction. Its velocity becomes just horizontal in two seconds after start.
Determine the velocity (by magnitude and direction) on start if the horizonal component of velocity is known to be $v_x=5\,\mathrm{m/s}$.

Solution

Let the vertical component of velocity be denoted by v_{0y}. It must have an upward direction, otherwise a horizontal velocity would never be reached. Write the vertical resolution of the theorem of change of linear momentum with a positive sense pointing up (it is shown by the label written at the beginning of the equation):

$$\uparrow: m\cdot(v-v_0)=(-m\cdot g)\cdot t \rightarrow 25\cdot(0-v_{0y})=-25\cdot 9.81\cdot 2$$

The solution is

$$v_{0y}=19.62\,\mathrm{m/s}$$

Once both its horizontal and vertical components are known, the magnitude of the initial velocity

$$v_0=\sqrt{v_x^2+v_{0y}^2}=\sqrt{5^2+19.62^2} \quad \rightarrow \quad v_0=20.25\,\mathrm{m/s}$$

The direction is specified by the angle it makes with the horizontal axis:

$$\tan\alpha=\frac{|v_{0y}|}{|v_x|}=\frac{20.25}{5} \quad \rightarrow \quad \alpha=76.13°$$

Exercise 2

A material particle of mass $m=80\,\mathrm{kg}$ moves in a horizontal plane xy under the action of a of a single constant force $\underline{F}$. The velocity in the time instant $t=2$ equals $v_0=\begin{bmatrix}+10\\-8\end{bmatrix}$ m/s, while the velocity is $v=\begin{bmatrix}-10\\+8\end{bmatrix}$ m/s at $t=8\,\mathrm{s}$.

Determine force $\underline{F}$.

Solution

Knowing both the initial and final velocities, write equations of the theorem of change of linear momentum in direction of axis *x*, as well as axis *y*:

$m\cdot(v_x - v_{0x}) = F_x(t - t_0) \rightarrow$

$m\cdot(v_y - v_{0y}) = F_y(t - t_0) \rightarrow$

Its solution gives the components of fthe force under consideration:

$F_{0x} =$

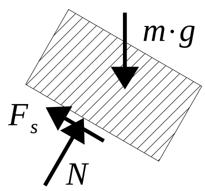
$F_{0y} =$

which can be arranged even in a vector as: $\underline{F} = \begin{bmatrix} \quad \end{bmatrix}$

Example 3

A body of $m = 10\,\mathrm{kg}$ is released from rest at the top of a slope of inclination $\alpha = 25°$. The body starts sliding down with a uniform acceleration and reaches a velocity of $10\,\mathrm{m/s}$ in 5 seconds. Calculate the coefficient of kinetic friction between the slope and the body.

Solution



All forces acting on the body are sketched in the figure. The body slides downwards, and force F_f equals therefore $F_f = \mu\cdot N$.

Write the theorem of change of linear momentum in a direction perpendicular to the slope: $10\cdot(0-0) = (N - 10\cdot 9.81\cdot\cos 25°)\cdot 5$

which yields the normal force component: $N = 88.91\,\mathrm{N}$

The friction force is then $F_f = \mu\cdot 88.91$

The theorem of change of linear momentum is also written in the direction of the motion:

$$10\cdot(10-0) = (10\cdot 9.81\cdot\sin 25° - \mu\cdot 88.91)\cdot 5 \quad ,$$

which solves to $\boxed{\mu = 0.2414}$

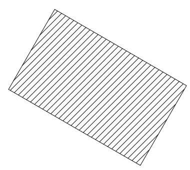
Remark 1: Application of the theorem in normal direction is justified rather by educational purposes: knowing the (zero) value of acceleration, Newton's second law should have rather been used in order to get the force *N*.

Remark 2: Repeating the same procedure in a parametric form it would be found that all terms of the last equation contained the mass on the first power. Since it could have been eliminated from the equation by division, it can finally be declared that the solution of the problem is independent of the mass of the body.

Exercise 3

A body of $m = 10\,\mathrm{kg}$ is released from rest at the top of a slope of inclination $\alpha = 25°$. The body starts sliding down with a uniform acceleration; the coefficient of (kinetic) friction is $\mu = 0.3$. What time is needed for the body to reach a velocity of $14\,\mathrm{m/s}$?

Solution



Make a sketch showing all forces that act on the body.

Write the theorem of change of linear momentum in a direction perpendicular to the motion:

This solves to

$$N=$$

Hence the force of friction is known as:

$$F_f=$$

Write the theorem of change of linear momentum in the direction of the motion:

Solving the equation, time is obteined as:

$$t=$$

Kinetic energy, work

Definition: Kinetic energy T of a material particle equals the half of product of its mass m and the square of its velocity v. Written in a formula, $T=\frac{mv^2}{2}$

The unit of work in SI is joule (abbreviated as J), it is a derived unit: $\mathrm{J}=\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2}$.

The following definition will concern the work done by a force. From earlier studies of physics, a definition like 'force times parallel translation' might sound familiar. Now another definition is given which will be equivalent to the former one in the case of a force of constant direction and magnitude, but will be cover cases when both the direction or magnitude of the force changes. It is introduced by defining first only a small work done on an elementary translation, which is not influenced by an occasional change in magnitude of the force.

Definition: Elementary work dL done on an elementary translation d$\boldsymbol{r}$ by force $\boldsymbol{F}$ acting on a material particle equals the dot product of the force and the elementary translation: $\mathrm{d}L=\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$.

Definition: The work L done along a given path $\boldsymbol{r}$ by force $\boldsymbol{F}$ acting on a material particle equals the sum of elementary works done by the force along the path (in a mathematically precise form, equals the integral of the force over the path): $L=\int \mathrm{d}L=\int_{\boldsymbol{r}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$.

Example 4

A body of mass $m=5\,\mathrm{kg}$ is pulled against the slope of inclination $\alpha=20°$. The force of traction is parallel to the slope and has a magnitude of $F=50\,\mathrm{N}$, the coefficient of (kinetic) friction is $\mu=0.2$. Calculate the work of forces acting on the body until it travels from the bottom to the top of the slope of length $s=13\,\mathrm{m}$.

Solution

Force F is parallel to the translation and they are directed uniformly, hence the work of F is:

$$L_F=+50\cdot 13=+650\,\mathrm{J}$$

The work done by gravity can be considered in two alternative ways. The resolution of weight along the translation is $s_g=s\cdot\sin\alpha=4.446\,\mathrm{m}$. The force is directed downwards, opposed to the translation, hence this work is negative:

$$L_g = -m \cdot g \cdot s_g = -5 \cdot 9.81 \cdot 4.446 = -218.1\,\mathrm{J}$$

Another possibility is to resolve the force in components in directions parallel and perpendicular to the slope. This latter component is also perpendicular to the translation, hence its work is zero. The magnitude of the parallel component is $m \cdot g \cdot \sin\alpha = 16.78\,\mathrm{N}$, it acts against the translation and does therefore a negative work being equal to the total work of weight:

$$L_g = -16.78 \cdot 13 = -218.1\,\mathrm{J}$$

Out of the two other forces exerted on the body by the slope, *N* is perpendicular to the surface and so to the translation, making its work to be zero:

$$L_N = 0 \quad .$$

In order to find the work of the friction, the magnitude of F_f should be obtained from the normal force *N*. It can be done using Newton's second law in the direction perpendicular to the surface, since there is neither translation nor acceleration in that sense:

$$m \cdot g \cdot \cos\alpha - N = m \cdot 0 \rightarrow N = m \cdot g \cdot \cos\alpha = 5 \cdot 9.81 \cdot \cos 20° = 46.09\,\mathrm{N}$$

From this, friction force $F_f = \mu \cdot N = 0.2 \cdot 46.09 = 9.218\,\mathrm{N}$ is obtained; its sense is opposed tothe motion, doing therefore a negative work:

$$L_f = -9.218 \cdot 13 = -119.8\,\mathrm{J}$$

Exercise 4

A body of mass $m = 15\,\mathrm{kg}$ slides down a slope of inclination $\alpha = 25°$.
The body is braked by a force $F = 50\,\mathrm{N}$ that is parallel with the slope.
The coefficient of (kinetic) friction is $\mu = 0.2$.
Calculate the work of forces acting on the body until it covers a distance $s = 6\,\mathrm{m}$.

Solution

The work done by force *F* (force magnitude times the translation parallel to it):

$$L_F =$$

The work done by gravity can be considered in two alternative ways. One approach is based on the works of components: the component perpendicular to the slope is also perpendicular to the translation and hence does no work, while the parallel component measures

$$m \cdot g \cdot \sin\alpha =$$

and does a work

$$L_g =$$

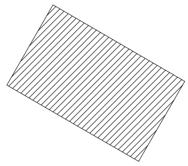
Another approach uses the component of the translation parallel to the force; the vertical component of the translation is

$$s_g =$$

on which the work is done by the weight in an amount of

$$L_g =$$

In order to calculate the work of all forces exerted by the slope on the body, the forces should be known first. Draw all forces acting on the body into the sketch and write a force resolution perpendicular to the slope:



$$\sum F_{i\nearrow}:$$

It can be solved for the normal force component and the friction force is obtained afterwards as:

$$N= \qquad\qquad F_f=\mu\cdot N=$$

The work done by the normal force is

$$L_N=$$

The work done by the force of kinetic friction is

$$L_f=$$

Let it be noted in advance that when a static friction is dealt with, no translation occur between the surfaces and hence static friction force typically does no work on a particle.

Theorem of change of kinetic energy

Theorem: The change in kinetic energy of a particle moving along a given path equals the work performed along the path by forces acting on the particle. It is also known as the *work–energy theorem*.

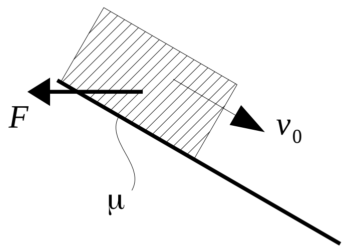
Written in a formula: $\frac{m}{2}\boldsymbol{v}_2^2-\frac{m}{2}\boldsymbol{v}_1^2=\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2}\boldsymbol{R}\cdot\mathrm{d}\boldsymbol{s}$, or briefly, $L_{1-2}=T_2-T_1$.

Usage

As clearly seen from the formula, there is neither acceleration nor time in it; consequently, the theorem can be used in any problems where the referred quantities do not appear either among given or requested parameters.

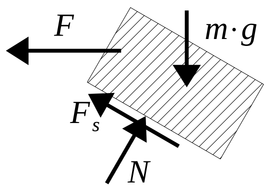
Example 5

A body of mass $m=15$ kg slides down a slope of an inclination $\alpha=30°$, its initial velocity is $v_0=7$ m/s. The body is decelerated by a horizontal force $F=120$ N. Find the distance covered by the body until it stops (the coefficient of kinetic friction is $\mu=0.15$).



Solution

In order to evaluate the work performed by the forces, at least the force components doing any work should be known. For this reason, all forces acting on the body are displayed in the figure: the weight that points down, the force F that causes deceleration, the force N that pushes the surfaces together and the friction force F_f that acts against the translation.



The weight is $m \cdot g = 15 \cdot 9.81 = 147.2\,\mathrm{N}$

Applying Newton's second law in a direction perpendicular to the slope we get:

$$\sum F_{i\nearrow}: N - m \cdot g \cdot \cos\alpha - F \cdot \sin\alpha = 0 \rightarrow \qquad N = 147.2 \cdot \cos 30° + 120 \cdot \sin 30° = 187.5\,\mathrm{N} \quad .$$

With the help of this, friction force is obtained as

$$F_f = 0.15 \cdot 187.5 = 28.13\,\mathrm{N}$$

In order to find the work performed by the weight as well as by the force *F*, horizontal and vertical components of the translation s should be known: $s_y = s \cdot \sin\alpha, s_x = s \cdot \cos\alpha$.

In terms of the preceding expressions, the work-energy theorem can be expanded as follows:

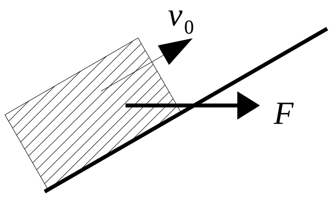
$$\frac{m}{2}(v^2 - v_0^2) = L_F + L_g + L_f + L_N$$

$$\frac{15}{2}(0^2 - 7^2) = -120 \cdot s \cdot \cos 30° + 147.2 \cdot s \cdot \sin 30° - 28.13 \cdot s + 0$$

It can be solved for *s* as $\boxed{s = 6.287\,\mathrm{m}}$

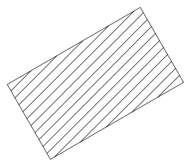
Exercise 5

A body of mass $m = 15$ kg is pulled up a slope of an inclination $\alpha = 30°$ by a horizontal force $F = 120\,\mathrm{N}$.
The velocity of the body is $v_0 = 3\,\mathrm{m/s}$ at the bottom of the slope.
Find the velocity of the body at the top of the slope of length of $10\,\mathrm{m}$? (The friction can be neglected.)



Solution

In order to evaluate the work performed by the forces, at least the force components doing any work should be known. For this reason, draw all forces acting on the body in the diagram.



Among them the weight is

$m \cdot g =$

Newton's second law written in a direction perpendicular ot the slope:

$\sum F \quad :$

which yields $N =$

In the calculation of work done by the weight and by force *F* one needs to know horizontal and vertical components of the oblique distance *s*:

$s_x =$ $\qquad\qquad s_y =$

Write the theorem of change of kinetic energy:

...and solve it for *v*:

$v=$

Example 6

An airplane travels westward at a height $h=1200\,\text{m}$, its velocity is horizontal and has a magnitude of $600\,\text{km/h}$. A paratrooper of mass $m=70\,\text{kg}$ jumps out of the airplane (its initial velocity is coincident to that of the airplane). At the instant of landing the velocity of the paratrooper has a magnitude of $1.2\,\text{m/s}$ and makes an angle of $80°$ with the horizontal. Calculate the total work of air resistance done on the paratrooper.

Solution

Because of the horizontal initial velocity, the trajectory of the paratrooper will be curvilinear. Since, however, no instantaneous accelerations or forces are given or asked in the problem, there is no need to know the trajectory. With the use of initial and final velocities, kinetic energy in both the initial and final position of the trajectory:

Initial velocity is $v_0=\frac{600}{3,6}=166.7\,\text{m/s}$, so $T_0=\frac{70\cdot 166.7^2}{2}=972611\,\text{J}$

On landing, $T=\frac{70\cdot 1.2^2}{2}=50.4\,\text{J}$

The force of gravity is $m\cdot g=70\cdot 9.81=686.7\,\text{N}$

In order to calculate its work, it is only possible now to use the component of translation parallel to the force (and not vice versa, since the trajectory that force resolution could be parallel to is not known):

$L_g=m\cdot g\cdot h=+686.7\cdot 1200=824040\,\text{J}$

The theorem of change of kinetic energy reads now as follows:

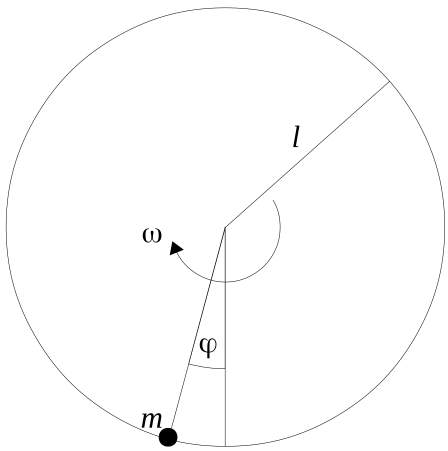
$T-T_0=L_g+L_e \;\rightarrow\; 50.4-972611=824040+L_e$

that can be solved to $\boxed{L_e=-1,796\cdot 10^6\,\text{J}}$.

Exercise 6

A body of mass $M=10\,\text{kg}$ is whirled on a chord of length $l=1.3\,\text{m}$ in a vertical plane. At a given instant the chord encloses an angle of $\varphi=10°$ with the vertical direction and moves with $v=15\,\text{m/s}$ just having left the bottom point.

Find the velocity of the body when it passes the top point.



Solution

The problem could (probably) be solved using Newton's second law as well, but in such a solution, instantaneous velocities should be found from variable accelerations in both normal and tangential sense which is rather a complicated case. Just for this reason, similar problems with a continuously changing state of motion are predominant in the use of theorems of change.

The particle is acted upon by two forces in the whole course of the motion; one of them is the weight:

$$m \cdot g =$$

The other is the cable force of variable magnitude and direction. This latter one is always directed radially towards the centre of the circle, so its elementary work done on the particle is

$\mathrm{d}\,L_S =$, hence its total work is $L_S =$.

The work of gravity is calculated for convenience by the component of translation aligned with the force, this component is

$$s_y =$$

The theorem of change of kinetic energy reads now as follows:

which can be solvec for *v* as: $v =$

Conservative forces

There exist some kinds of forces whose work done on a moving particle is found to be independent of the geometry of the path the work is performed along: this work depends only on the initial and final position of the particle. These forces are called conservative forces and can be assigned with a potential function *U* dependent on the position of the particle, whose change between two points equals the negative of the work done by the forces on the particle when it is moved from one point to another: $U_2 - U_1 = -L_{1-2} \rightarrow L_{1-2} = U_1 - U_2$.

The word 'potential' refers to capability, a potential function describes the ability of a force to perform work. If a force did work indeed, then its ability to do further work is decreased in the same amount.

Since a potential function is time-independent, coincident initial and final points of a path imply zero work without respect to the geometry of the path. In a concise form: conservative forces do zero work along any closed path.

Examples

The *force of gravity* is a conservative force: if *h* denotes height above an arbitrary base level, the potential function of gravity can be written in the form $U(h) = m \cdot g \cdot h$. During lifting a particle from the base level to height *h*, the work performed by gravity equals $-m \cdot g \cdot h$ (the force acts downwards, the translation *h* is directed upwards, that is why the sign is negative), as far as the potential function yields the same conclusion by the formula $U_1 - U_2 = 0 - m \cdot g \cdot h$.

The *spring force* arising in a linear spring (understood as exerted by the spring on the connected body) is a conservative force. Let the factor of proportion between spring force and elongation be denoted by *k*, then the potential function of the force exerted by a spring on the connected body at an elongation Δl is $U(\Delta l) = \frac{k \Delta l^2}{2}$.

Counterexample: The force of friction acts always against translation and is therefore able to do negative work only (or no work at all). Consequently, if there is any work done by friction along a closed path, the total work cannot be zero: the force of friction is a nonconservative force.

Mechanical energy

Definition: The sum of kinetic energy of a material particle and the potential function of all forces acting on that particle is called mechanical energy (the usual notation for this sum is $T+U$).

Theorem: If all forces performing work on a moving particle are conservative, then the mechanical energy is kept constant during motion. It is also known as the *theorem of conservation of mechanical energy.*

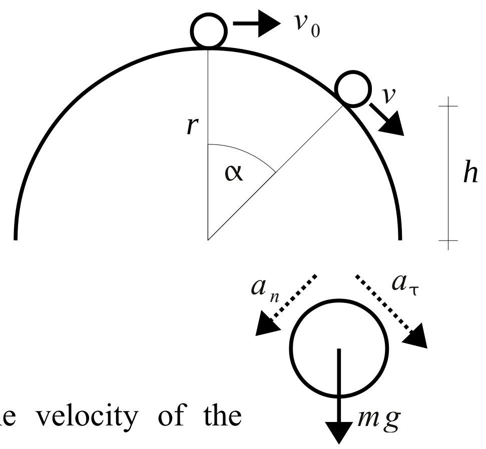
It can also be written in the concise form $T+U=\text{constant}$ but in fact, the equality between energies at the beginning and at the end of motion is mostly written. For example, in the case of a motion from point 1 to 2 the equation $T_1+U_1=T_2+U_2$ is written down and solved for some unknown.

(The proof of a theorem is also based on this principle: stating the theorem of change of kinetic energy between any two points in the form $T_2-T_1=L_{1-2}$, the work of conservative forces can be plugged in: $T_2-T_1=U_1-U_2$. After an arrangement we have $T_1+U_1=T_2+U_2$, and since points 1 and 2 may denote initial and final points of any segment of the trajectory of the motion, the sum of the two types of energy must remain constant.)

Example 7

A material particle of mass m starts sliding with an initial velocity v_0 from the top point of a hemisphere of radius r. Find the height h where the particle leave the surface.

Solution



Both initial and final positions are shown in the figure to the right. The height in terms of angle α is $h=r\cos\alpha$.

When the particle is detached, no forces are exerted on it by the surface: the only force is from gravity, which generates accelerations on the spherical path alone. Writing Newton's second law in normal direction:

$$m\,g\cos\alpha=m\,a_n=m\frac{v^2}{r} \quad ,$$

which yields $r\cos\alpha=\frac{v^2}{g}$ after simplification. It lets the velocity of the particle to be expressed as $v=\sqrt{h\cdot g}$ when it leaves the sphere.

Gravity is the only force that performs work between initial and final configurations. Since it is conservative, the theorem of conservation of mechanical energy applies. Kinetic energies in the initial and final configurations are written as $T_0=\frac{1}{2}m\cdot v_0^2$ and $T=\frac{1}{2}m\cdot v^2$, respectively.

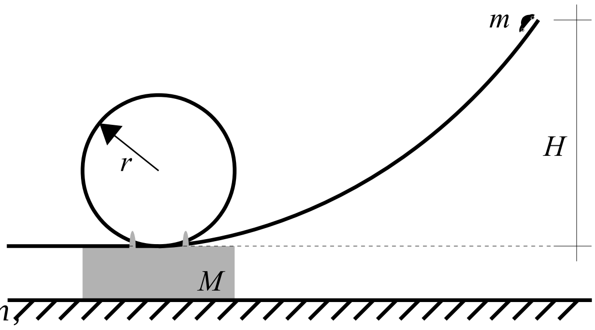
Let the height of the particle be related to the centre of the sphere; thus, the potential energies in the initial and final configurations read $U_0=m\cdot g\cdot r$ and $U=m\cdot g\cdot h$, respectively. According to the theorem,

$$T_0+U_0=T+U\rightarrow\frac{1}{2}m\cdot v_0^2+m\cdot g\cdot r=\frac{1}{2}m\cdot v^2+m\cdot g\cdot h \quad .$$

After simplification by the mass, multiplication by two and plugging the velocity of leaving the sphere in, the equation $v_0^2+2\,g\cdot r=g\cdot h+2\,g\cdot h$ is obtained that can be solved for h:

$$\boxed{h=\frac{2}{3}r+\frac{v_0^2}{3\,g}} \quad .$$

Exercise 7



A matchbox released at a height H follows the given track. The car passes the loop of radius r in a way that is never released from the track and the foundation is not moved by the car either.

a) Find the velocity v the car has by passing the top of the loop in function of H.

b) Find the limits for height H.

The matchbox is modelled as a particle of mass m, the foundation of the loop is of mass M, friction as well as other masses can be neglected.

$r=15\,\text{cm},\ m=0.05\,\text{kg},\ M=0.5\,\text{kg}$

Solution

The matchbox starts from rest, hence its initial velocity and kinetic energy are:

$$v_0= \qquad\qquad\qquad T_0=$$

On the top of the loop, kinetic energy as a function of the unknown velocity v is:

$$T=$$

Here the use of the theorem of change of kinetic energy seems to be a solution of general scope. However, by neglecting friction, the net force exerted on the car by the track will always be perpendicular to the track, making the normal force component unable to do work. In this case only gravity performs work: since it is conservative, the theorem of conservation of mechanical energy applies. Let the base level of the potential of weight be chosen to fit the lowest point of the track.

At an initial time instant, the height of a car above the base level is , so its potential energy at the same time is

$$U_0=$$

On the top of the loop, the height of a car above the base level is , so its potential energy at the same time is

$$U=$$

Write the theorem with these data:

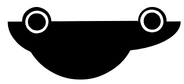
$T_0 + U_0 = T + U \rightarrow$

The square of velocity (and then the velocity itself) can be expressed from the above formula:

$v^2 =$

a) $v =$

When passing the top of the loop, the car moves along a circular path. Draw all forces acting upon it in the figure. (Since it touches the track, two forces are exerted on it.)



How large is the acceleration generated by those forces?

$a_n =$

Write Newton's second law along the direction of acceleration, then plug the expression obtained for the velocity into it :

$m \cdot g + N = m \cdot a_n \rightarrow$

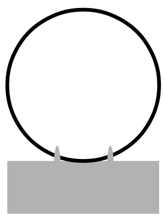
Now express force *N* from the equation:

$N =$

This force must satisfy two conditions.

Since the car keeps in contact with the track, the force between them cannot be tensile (in this case the car would fall down). Stating and then solving the corresponding inequality for *H* we have:

Another condition is that the foundation cannot be released from the ground. This part should remain in equilibrium while acted upon by three forces as follows: its weight, the negative of the force exerted upon the car and a support force (let it be denoted by *F*) transmitted by the ground. Their equilibrium is expressed as



$\sum F_{iy}:$

Now substitute the earlier expression for force *N* and express force *F* from it:

The foundation will not be released if force *F* is compressive. Stating and then solving the corresponding inequality for *H* we have:

Finally, two unified conditions together can be written as follows:

b) $< H <$

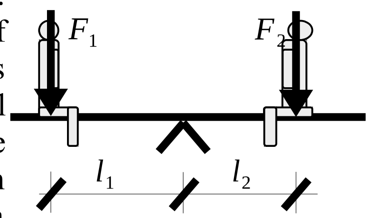
Moment of a force

In the precedeing lectures material particles were only dealt with, ignoring the extension of the body as well as the location of forces. It is obvious by the example fo a merry-go-round for anybody that a force can result not just in translation but also rotation of a body. This turning effect is called the *moment* of a force and denoted commonly by *M*.

Definition: The moment of a force about a given axis is calculated as a product of the magnitude of the perpendicular component of the force (to the axis) and the arm of that force component. This *moment arm* is defined as the (perpendicular) distance between the given axis and the line of action of the force. The moment is positive if the sense of its rotation is counterclockwise viewed down in front of the arrow of the given axis.

Definition: Within a plane, the moment of a force about a point equals the moment of the same force about an axis directed perpendicularly in front of us which passes through the point in case. A moment is positive if rotates counterclockwise.

Perhaps the simplest illustration of the effect of moment is the seesaw: considering the problem in plane, the moments about the pinpoint of the seesaw are calculated. A necesary condition of the equilibrium is that moments of two forces on opposite sides of the seesaw be of equal magnitude and opposite sense. The weight of a person sitting at one side of the axis (pinpoint) has a moment opposite to that of sitting on the other side. Moments in both cases can be found as the product of a weight multiplied by its distance from the axis, that is,



$$F_1 \cdot l_1 = F_2 \cdot l_2 \quad .$$

is needed for equilibrium, showing that a larger force F_2 implies a smaller moment arm l_2 .

Of course, more than two such rotating effects can be found in real systems, therefore, the condition of equilibrium is not defined in general through an equality of two effects, but their signed sum is made equal to zero. For instance, considering the counterclockwise sense to be positive, then the condition of equilibrium reads $F_1 \cdot l_1 - F_2 \cdot l_2 = 0$. This, if holds, can easily be arrranged to the equality of two moments; but if not, then it can be seen from the sign of the left hand side which sense the system starts to rotate in (if the sign is positive, a counterclockwise rotation will occur).

We remark that horizontal bar shown in the figure is acted on by one force more, exerted by the pin in an upright vertical direction (denoted it by *T*). Since its line of action passes through the axis of rotation, its arm is zero, making the corresponding term excluded from our equation. Equilibrium, however require not only the sum of moments but also the sum of force components along any direction to be zero, hence the vertical resolution equation is $T - F_1 - F_2 = 0 \rightarrow T = F_1 + F_2$. Moreover, the body should not only be balanced against rotation about the pinpoint but also, e.g., about the point of application of F_1. The arms of forces F_1., T and F_2 with respect to this point are 0, l_1 and l_1.+ l_2, respectively, and hence the moment equilibrium equation can be written as $F_1 \cdot 0 + (F_1 + F_2) \cdot l_1 - F_2 \cdot (l_1 + l_2) = 0$ that yields the same condition $F_1 \cdot l_1 = F_2 \cdot l_2$ as got earlier.

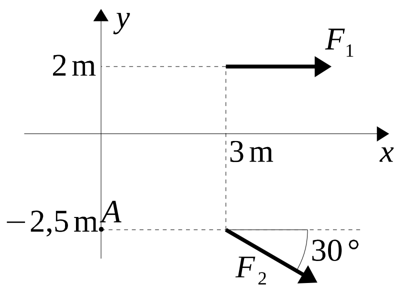
In order to skip repeated references to the clock, the moment will be considered positive (unless stated otherwise) in a moment equation if it rotates counterclockwise. Some rules to consider:

- This moment is due to the force. If the same force is resolved (decomposed) into components in any point on its line of action, then the sum of moments of components add to the moment of the force (as componenets also add to the force itself); this rule is known as Varignon's theorem.

- If the line of action of a force passes through the axis (or point in 2D) of a moment, it does not rotate about the axis (point).
- A force component parallel to the axis of moment does not rotate about the axis either. This follows obviously from the definition of moment but considering parallel lines as those intersecting at infinity, it also follows from the two preceding rules.
- Positive or negative sense is decided by the handedness of coordinate system; that is, if a left-handed system is used in some literature, then clockwise moment has the positive sense.

Example 1

Calculate the moment of forces F_1 and F_2 shown in the figure about point A as well as about the origin.
$F_1 = 12\,\mathrm{kN}, F_2 = 9\,\mathrm{kN}$



Solution

Moment of force F_1

The force rotates clockwise about the origin, thus, its sense is negative. The moment arm equals the vertical distance of the line of action of F_1 and the origin (2m):

$$M_1^{(0)} = -12 \cdot 2 = -24\,\mathrm{kNm}.$$

The force rotates clockwise about point A, it is therefore negative; the arm equals the vertical distance 2+2,5 = 4,5m:

$$M_1^{(A)} = -12 \cdot 4{,}5 = -54\,\mathrm{kNm}.$$

Moment of force F_2

A momenst of an oblique force can be calculated as the sum of moments of component forces along *x* and *y*. These components are

$$F_{2x} = +9 \cdot \cos 30° = +7.794\,\mathrm{kN}(\rightarrow), \quad F_{2y} = -9 \cdot \sin 30° = -4.5\,\mathrm{kN}(\downarrow).$$

With a decomposition done in the given point of application, the horizontal components rotates counterclockwise about the origin and is positive, its arm is the distance measured from axis *x* (2.5 m); whereas the vertical component rotates clockwise (it is negative), its arm is the distance measured from axis *y* (3 m):

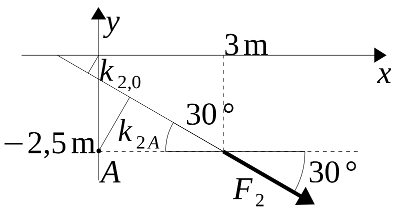
$$M_2^{(0)} = +7.794 \cdot 2.5 - 4.5 \cdot 3 = +5.985\,\mathrm{kNm}(\curvearrowleft)$$

Observe that after the senses of moments have been decided by inspection, only absolute values of forces and distances are accounted for.

Using the same point for decomposing the force, the horizontal one does not rotate about *A* (its arm is zero), the vertical component rotates in negative sense (clockwise) with an arm equal to the horizontal distance (3 m):

$$M_2^{(A)} = +7.794 \cdot 0 - 4.5 \cdot 3 = -13.5\,\mathrm{kNm}(\curvearrowright)$$

Moments could also be calculated without resolving the force into components but it needs some



careful preliminary steps concerning geometry as follows:

In order to get the distance between the line of action and the origin, the point of intersection between that line of action and axis y is located first. It is at a height of $3 \cdot \mathrm{tg}\, 30° = 1.732\ \mathrm{m}$ above point A, thus, at a depth of

$2.5 - 1.732 = 0.768\,\mathrm{m}$ below the origin. Considering the right triangle fitted to the origin:

$$k_{20} = 0.768 \cdot \cos 30° = 0.665\,\mathrm{m}\quad .$$

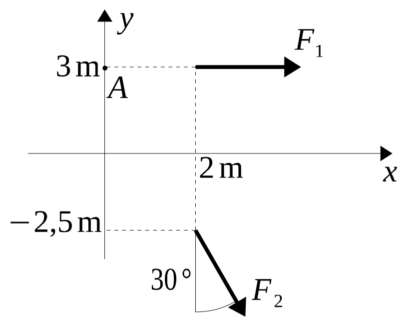
Concluded from the position of line of action of F_2 , its moment is positive (counterclockwise): $M_2^{(0)} = +9 \cdot 0.665 = +5.985\,\mathrm{kNm}\,(\curvearrowleft)$, exactly as obtained from components.

The distance of the same line of action from point A is:

$$k_{2A} = 3 \cdot \sin 30° = 1.5\ \mathrm{m}$$

Looking at the prolonged line of action of F_2 about point A, it rotates counterclockwiseand has therefore a positive moment: $M_2^{(A)} = -9 \cdot 1.5 = -13.5\ \mathrm{kNm}\,(\curvearrowright)$.

Exercise 1



Calculate the moment of forces F_1 and F_2 shown in the figure about point A as well as about the origin.
$F_1 = 9\,\mathrm{kN}, F_2 = 12\,\mathrm{kN}$

Solution

Moment of force F_1

How does the force rotate about the origin compared to the clock?

.... , hence its sign is ....

Distance between the line of action and the origin (moment arm) is ....

The moment of the force: $M_1^{(0)} =$ ....·.......=...... kNm( )

Moment about point A:

$M_1^{(A)} =$ ....·.......=...... kNm( )

Moment of force F_2

The force is oblique, consider rather by perpendicular components (through any point in its line of action) instead of by oblique resultant and distance:

$F_{2x} =$ ....·...........=..................... kN( ),
$F_{2y} =$ ....·...........=..................... kN( ).

When calculating moment about the origin:

Sense of rotation of the horizontal component: ......, sign of moment: ....., moment arm: ....m.

Sense of rotation of the vertical component: ......, sign of moment: ....., moment arm: ....m.

The moment of the force: $M_2^{(0)} = \ldots\ldots\ldots\cdot\ldots\ldots\ldots \quad \ldots\ldots\ldots\cdot\ldots\ldots\ldots = \ldots\ldots \text{ kNm}(\quad)$

When calculating moment about point A:

Sense of rotation of the horizontal component: ......, sign of moment: ....., moment arm: ....m.

Sense of rotation of the vertical component: ......, sign of moment: ....., moment arm: ....m.

The moment of the force: $M_2^{(A)} = \ldots\ldots\ldots\cdot\ldots\ldots\ldots \quad \ldots\ldots\ldots\cdot\ldots\ldots\ldots = \ldots\ldots \text{ kNm}(\quad)$

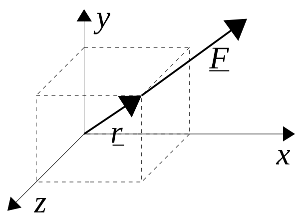
Calculation of moments in 3D can be reduced to a planar problem by considering only components and projected distances that are perpendicular to the axis of moment.

A moment taken about a point in space is a vector whose component along any axis t equals the moment taken about an axis t' which is parallel to t and passes through the point under consideration. Consequently, if moments about each axis passing through that point and being parallel to one of the coordinate axes are arranged in a vector, it will exactly be the vector of moment about the point in case.

Example 2

Calculate the moment of force $\underline{F}$ applied at a point given by vector $\underline{r}$ about axes x, y, z as well as about the origin.

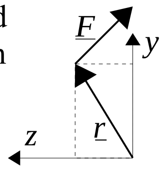
$$\underline{F} = \begin{bmatrix} 5 \\ 7 \\ -6 \end{bmatrix} \text{kN}, \quad \underline{r} = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} \text{m}$$



Solution

When a moment about x is looked for, the problem should be viewed in front of the axis, i.e., frok right to left (see the figure aside). Both components rotate in negative sense:

$$M_x = -7\cdot 3 - 6\cdot 8 = -69 \text{ kNm}(\curvearrowright)$$

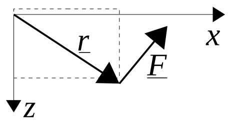


$F_y = 7 \text{ kN}(\uparrow)$
$F_z = -6 \text{ kN}(\rightarrow)$
$r_y = 8 \text{ m}$
$r_z = 3 \text{ m}$

The sign of moment should here be interpreted as seen in front of $+x$.

Finding moment about axis y requires an opposite view (i.e., a top view) as shown in the attached figure. Both components rotate in a positie sense, thus:

$$M_y = +5\cdot 3 + 6\cdot 4 = +39 \text{ kNm}(\curvearrowleft)$$

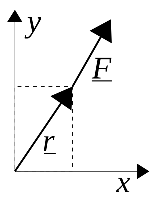


$F_x = 5 \text{ kN}(\rightarrow)$
$F_z = -6 \text{ kN}(\uparrow)$
$r_x = 4 \text{ m}$
$r_z = 3 \text{ m}$

The sign of moment is now interpreted as seen in front of $+y$.

Finally, in order to get moment about axis z, a top view as shown again in the attached figure is needed. Horizontal and vertical components rotate in a negative and positie sense, respectively:

$$M_z = -5\cdot 8 + 7\cdot 4 = -12 \text{ kNm}(\curvearrowright)$$



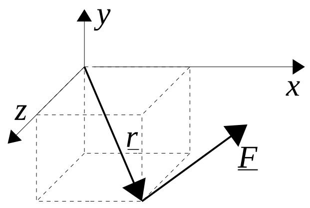
$F_x = 5 \text{ kN}(\rightarrow)$
$F_y = 7 \text{ kN}(\uparrow)$
$r_x = 4 \text{ m}$
$r_y = 8 \text{ m}$

A nyomaték előjele itt a z tengely irányából nézve értendő.

From the three above components, the moment of F about the origin is $\underline{M}_F^0 = \begin{bmatrix} -69 \\ +39 \\ -12 \end{bmatrix} \text{kNm}$.

Exercise 2

Calculate the moment of force $\underline{F}$ applied at a point given by vector $\underline{r}$ about axes x, y, z as well as about the origin.



$$\underline{F}=\begin{bmatrix}8\\4\\-3\end{bmatrix}\text{kN},\ \underline{r}=\begin{bmatrix}5\\-7\\3\end{bmatrix}\text{m}$$

Solution

In order to find moment about axis *x*, sketch a view in front of the same axis *x*:

Decide the sensse of rotation as well as the sign of both components based on the sketch.

$F_{...}=.....\,\text{kN}(...)$
$F_{...}=.....\,\text{kN}(...)$
$r_{...}=.....\,\text{m}$
$r_{...}=.....\,\text{m}$

$$M_x=.......\cdot.......\quad.......\cdot.......=\quad........\text{kNm}$$

Next. in order to find moment about axis *y*, sketch a view in front of the same axis *y*:

Decide the sensse of rotation as well as the sign of both components based on the sketch.

$F_{...}=.....\,\text{kN}(...)$
$F_{...}=.....\,\text{kN}(...)$
$r_{...}=.....\,\text{m}$
$r_{...}=.....\,\text{m}$

$$M_y=.......\cdot.......\quad.......\cdot.......=\quad........\text{kNm}$$

Finaly, when the moment about axis *z* is looked for, sketch a view in front of the same axis *z*:

Decide the sensse of rotation as well as the sign of both components based on the sketch.

$F_{...}=.....\,\text{kN}(...)$
$F_{...}=.....\,\text{kN}(...)$
$r_{...}=.....\,\text{m}$
$r_{...}=.....\,\text{m}$

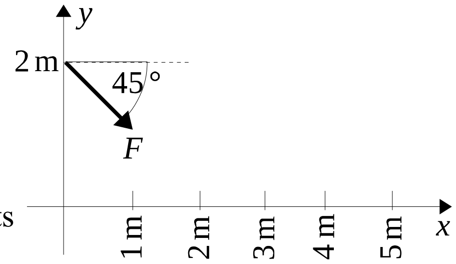
$$M_z=.......\cdot.......\quad.......\cdot.......=\quad........\text{kNm}$$

Each of there moment components are moments calculated about axes passing through the origin; consequently, the vector of moment about the origin is

$$\underline{M}^{(0)}=\begin{bmatrix}M_x\\M_y\\M_z\end{bmatrix}=\begin{bmatrix}\quad\\\quad\\\quad\end{bmatrix}\text{kNm}\ .$$

Example 3

Find moments of force $F=14\,\text{kN}$ about each point given on axis x.



Solution

Because of the angle 45°, vertical and horizontal components of force *F* differ only in direction but not magnitude, that is,

$$F_x = F_y = 14 \cdot \frac{\sqrt{2}}{2} = 9.899\ \text{kN}$$ to the right and downwards.

If the force is resolved into components in its point of application, the horizontal component rotates in a negative sense about each point with an arm of 2 m; the vertical component rotates in a positive sense about each point with an arm equal to the coordinate x of each point. The moments are

$$M_1 = -9.899 \cdot 2 + 9.899 \cdot 1 = -9.899\ \text{kNm}\,(\curvearrowright)$$

$$M_2 = -9.899 \cdot 2 + 9.899 \cdot 2 = 0\ \text{kNm}$$

$$M_3 = -9.899 \cdot 2 + 9.899 \cdot 3 = +9.899\ \text{kNm}\,(\curvearrowleft)$$

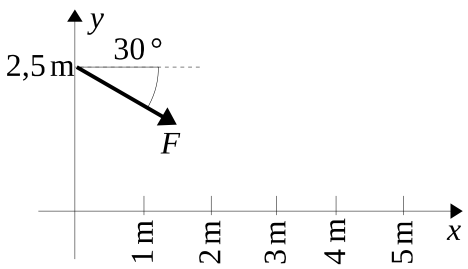
$$M_4 = -9.899 \cdot 2 + 9.899 \cdot 4 = +19.80\ \text{kNm}\,(\curvearrowleft)$$

$$M_5 = -9.899 \cdot 2 + 9.899 \cdot 5 = +29.70\ \text{kNm}\,(\curvearrowleft)$$

Remark: zero value of M_2 means that force F does not rotate about that pointl. It can only happen if its line of action passes through the point at 2 m which is the case indeed.

Exercise 3

Find moments of force $F = 16\,\text{kN}$ about each point given on axis x.



Solution

The moment of an oblique force can be obtained as a sum of moments of its components. Let the resolution into components be done in the point of application of F:

F_x =........·..........=.................... kN(   ),

F_y =........·..........=.................... kN(   ).

The horizontal component rotates about the given points ...., hence the sign of its moments is ...., and its moment arm measures......................................

The vertical component rotates about the given points ...., hence the sign of its moments is ...., and its moment arm measures......................................

In the light of this, moments of the force about marked points are:

$M_1 =$ $\qquad\qquad\qquad\qquad$ =

$M_2 =$ $\qquad\qquad\qquad\qquad$ =

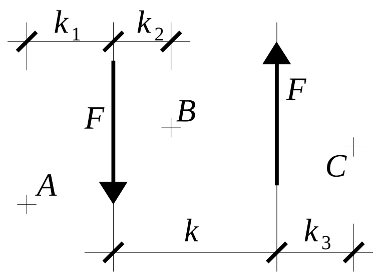
$M_3 =$ $\qquad\qquad\qquad\qquad$ =

$M_4 =$ $\qquad\qquad\qquad\qquad$ =

$M_5 =$ $\qquad\qquad\qquad\qquad$ =

Couples

It has already been seen that two forces of equal magnitude and opposite sense are equivalent to the zero force if they share a common line of action. Two forces are called to represent a couple if they have the same magnitude, opposite sense and their lines of action are parallel but not coincident. If resolution equations are written in order to calculate the resultant of a couple, all components of both forces in the couple will appear with the same magnitude and opposite sign, adding up all to zero. Moment is, however, nonzero: its magnitude is the same about all points. It can also be proved by the cross-product definition of the moment of a force, but it is shown here only by a scalar approach within the plane of two parallel forces, see the figure below for illustration:



The sum of moments about point A, B and C, respectively:

$$M_A = -F\cdot k_1 + F\cdot(k+k_1) = F\cdot k$$

$$M_B = F\cdot k_2 + F\cdot(k-k_2) = F\cdot k$$

$$M_C = F\cdot(k+k_3) - F\cdot k_3 = F\cdot k$$

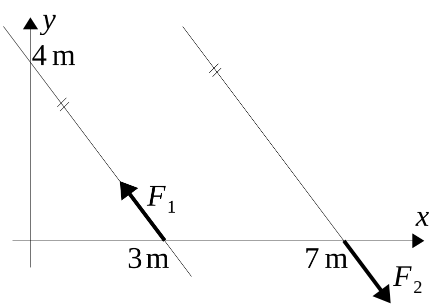
Thus, the two force generates the same moment about any point; their resultant is purely this rotating effect. This effect of rotation that a couple is equivalent to is called *torque*. The magnitude of a torque, as illustrated by the three equations above, depends on two conditions: the magnitude of forces in the couple and the distance between their line of action, called also the arm of the couple. The sense of rotation and hence the sign of a torque depends on the relative position of the two forces and the easiest decision can be made on it by simply checking the sense of moment caused by one force about any point on the line of action of the other.

Example 4

Determine the resultant of a couple given by forces $F_1 = F_2 = 12\,\text{kN}$ according to the figure

a) by summation of the moments of each force about the origin,

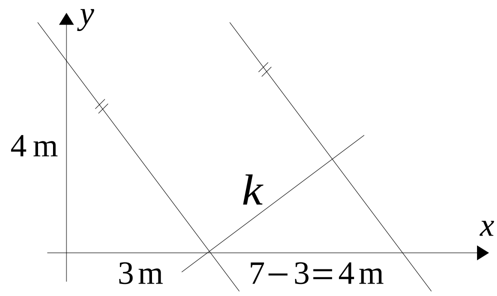
b) by calculating the arm of the couple.



Solution

The moment of two forces about the origin can be found in several ways. For a change, let F_1 and F_2 be resolved into components in their intersection with axis y and x, respectively; thus, only the horizontal component of F_1 and only the vertical component of F_2 rotates about the origin. The value of moment is as follows:

$$M=+12\cdot\frac{3}{\sqrt{3^2+4^2}}\cdot 4-12\cdot\frac{4}{\sqrt{3^2+4^2}}\cdot 7=-38.4\,\text{kNm}\,(\curvearrowright)$$



If the distance of two parallel lines is looked for, the easiest way is to compare two similar right triangles in the figure (see the longer legs and hypotenuses):

$$\frac{k}{4}=\frac{7-3}{\sqrt{3^2+4^2}}\rightarrow k=3.2\,\text{m}$$

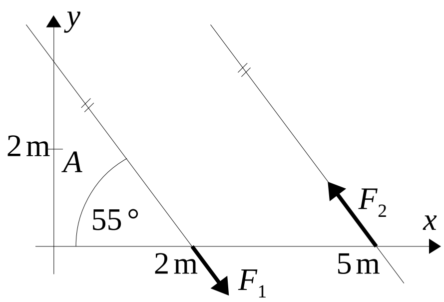
To obtain the sign of the moment, one needs to consider that force F_1 rotates counterclockwise about the point of application of F_2. It means that its moment is negative: $M=-12\cdot 3.2=-38.4\,\text{kNm}\,(\curvearrowright)$

Exercise 4

Determine the resultant of a couple given by forces $F_1=F_2=15\,\text{kN}$ according to the figure

a) by summation of the moments of each force about point A,

b) by calculating the arm of the couple.



Solution

Resolve both forces into components in their point of application, showing the sense of components as well:

$F_{1x}=$ $\qquad\qquad\qquad$ $,F_{1y}=$

$F_{2x}=$ $\qquad\qquad\qquad$ $,F_{2y}=$

The sum of moments of four forces about point A (sign, force magnitude, moment arm):

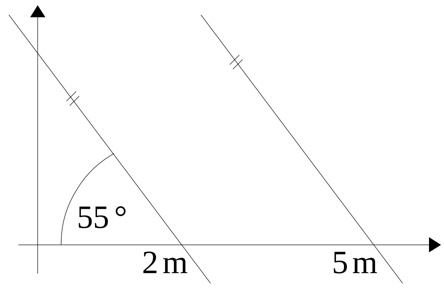
$$M^{(A)}= \;.....\cdot...\;\; .....\cdot...\;\; .....\cdot...\;\; .....\cdot...= \;.....\,\text{kNm}(\quad)$$

The moment arm equals the distance between parallel forces.

Draw that distance in the figure and look for similar triangles.

The moment arm is therefore

$$k= \qquad\qquad =$$



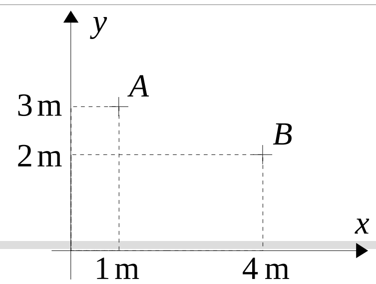
In which sense do forces rotate about each other?

The moment of the couple (sign, magnitude, moment arm) is:

$$M=..F\cdot k= \qquad\qquad =$$

Example 5

Replace the moment $M=27\,\text{kNm}\,(\curvearrowleft)$ by a couple of vertical forces passing through points A and B, respectively.



Solution

Two vertical lines through A and B lie at a distance of $k=3$ m defining the arm of the couple.

Magnitudes for A and B are obtained by the rearrangement of formula $M = F \cdot k$:

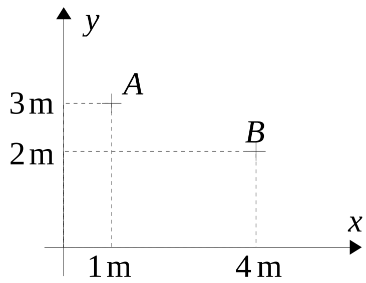
$$A = B = \frac{M}{k} = \frac{27}{3} = 9\,\mathrm{kN}$$

In order to set the arrows of forces correctly, the sense of rotations should be adjusted. A vertical force through point A rotates about B in a positive sense if directed downwards and in negative sense if directed upwards. The moment given in th problem corresponds to the former case; and the other force of the couple must have an opposite direction: B points therefore up: $\boxed{A = 9\,\mathrm{kNm}(\downarrow), B = 9\,\mathrm{kNm}(\uparrow)}$

Remark: since the forces here are all vertical, no y coordinates appeared in the calculations because of the free translation of forces in their own line of action.

Exercise 5

Replace the moment $M = 27$ kNm (↷) by a couple of horizontal forces passing through points A and B, respectively.



Solution

The arm of the couple equals the distance between two parallel lines of action of forces:

$$k = \ldots\ldots \mathrm{m}$$

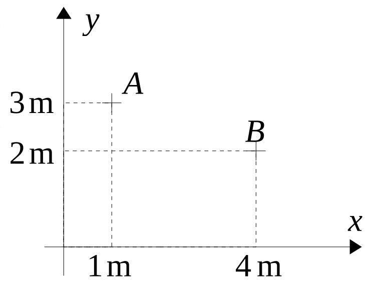
Using this value, the magnitude of forces in the couple is

$$|A| = |B| = \frac{|M|}{k} = \frac{\quad}{\quad} = \ldots\ldots\ldots \mathrm{kN}$$

What direction should the horizontal force A have in order to rotate about point B in the same sense as moment M does? ....

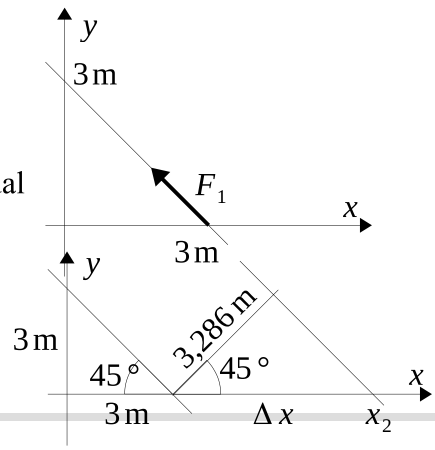
What direction should the horizontal force B have in order to rotate about point A in the same sense as moment M does? ....

Sketch the result.



Example 6

Replace the torque $M = -23$ kNm (↷) by a couple involving a force $F_1 = 7$ kN as shown in the figure.



Solution

One force in a couple is given, the other should have equal magnitude and opposite sense: $F_2 = 7\,\mathrm{kN}(\searrow)$.

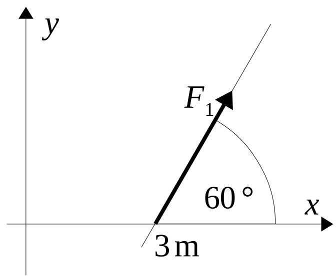
The moment of a couple will be of the desired magnitude if its arm is $k = \frac{23}{7} = 3.286\,\mathrm{m}$

This distance should be measured perpendicularly to the given line of action in order tho get the line of action of F_2 ; this can be done both to the right and upwards or to the left and downwards. With the first choice, F_2 rotates in a negative sense about the point of application of F_1 matching exactly the requirements of the problem. This translation of the line is shown in the figure, making possible to find the intersection of line of action of F_2 and axis *x* as follows:

$\Delta x = 3.286 \cdot \sqrt{2} = 4.647\,\text{m}$ that yields $x_2 = 3 + 4.647 = 7.647\,\text{m}$

Exercise 6

Replace the torque $M = +23\,\text{kNm}$ (↶) by a couple involving a force $F_1 = 8\,\text{kN}$ as shown in the figure.



Solution

What direction does force F_2 have? .....

How large is its magnitude? $F_2 = ...\,\text{kN}$

The arm of the couple can be obtained from the magnitudes of the torque and the forces:

$$k = \frac{M}{F} = \frac{\quad}{\quad} = ...$$

In which direction should F_2 be translated with respect to F_1 ? ....

What is the horizontal shift between the two lines of action? ....

Resultant of parallel force systems in a plane

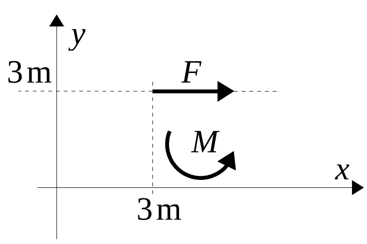
The *resultant* of a plane force system is a single force or moment having the same effect on a body.

When two forces share a line of action, their resultant is known to be a force still in the same line of action, and its signed magnitude is defined by the sum of (signed) magnitudes of the two forces (under a special choice of two equal forces opposed to each other, this is specifically the zero force). There have already been found examples at couples that if the lines of action do not coincide, it is not sufficient to use resolution equations for finding the resultant but rotational effect should also be accounted for.

Consider first what kind of effect is a compound of a force and a torque equivalent to.

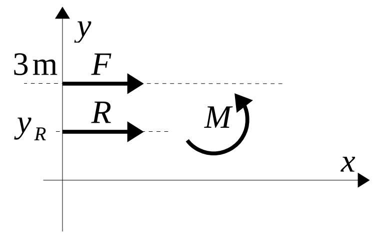
Example 7

Determine the resultant of the force $F = 13\,\text{kN}$ and torque $M = 14\,\text{kNm}$ given in the figure.

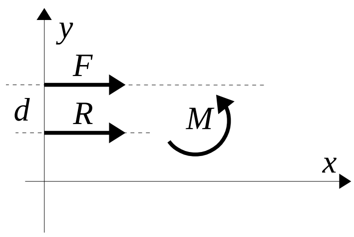


Solution

In order that a resolution equation can be evaluated, any torque *M* should be rewritten as a couple. At this stage of studies, however, it is known that two forces forming a couple would be of equal magnitude and opposite sense, making each other to be cancelled in any resolution equations. Thus, in the resolution equation of the problem currently addressed, only the force F will appear, making sure that the resultant has the same magnitude and direction as F have (it points to the right horizontally): $R=13\,\mathrm{kN}\,(\rightarrow)$. It is yet to be clarified where that resultant exactly is: it is only known that its line of action should be located in a way that R could rotate by the same moment about any point as the given force F and torque T together. The line of action of R can uniquely be specified by finding its intersection y_R made with axis y. It can be obtained by the moment equation written about the origin (the term in brackets is the value of R, already known): $\sum M_i^{(0)}:-13\cdot 3+14=-(13)\cdot y_r$, yielding $y_R=1.923\,\mathrm{m}$ (positive sign of this coordinate confirms that the intersection is above the axis x indeed).

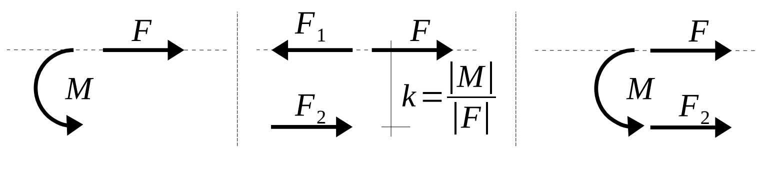


The line of resultant could also be specified by another parameter. Let d denote the the vertical shift of the resultant with respect to the line of F downwards. Equivalence of moments about the point of application of force F is written as $\sum M_i^{(F)}:+13\cdot 0+14=+13\cdot d$, hence $d=1.077\,\mathrm{m}$ (positive sign confirms here that our assumption on a lower position of R than of F is true).



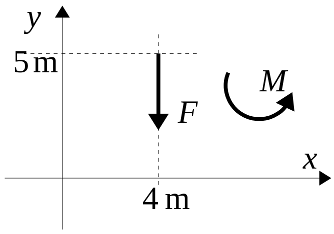
As a generalization of this latter method, one can say that, writing the equivalence of moments about the point of application of the force, only the torque on one side and only the moment of the force on the other side will be nonzero. This also means that the resultant must be located on the side of the given force where it results in a rotation of the same sense as the torque has. The distance between lines is given by the formula $d=\frac{|M|}{|R|}=\frac{|M|}{|F|}$. In summary, the resultant of a force and a torque is another force with the same direction and magnitude as the given force has, but its line of action is translated perpendicularly by a distance directly and inversely proportional to the magnitude of the torque and of the force, respectively.

The same problem can also be solved by a method only slightly different from the previous one. The torque M can be replaced by a couple composed of forces F_1 and F_2 . The resultant of force F and torque M is the same as that of F, F_1 and F_2 . Let the force F_1 be chosen to be the negative of F (even in a common line of action with F). Subtracting now the force system (F, F_1) from the initial system (F, F_1 , F_2), only a single force F_2 remains but the resultant does not change. This force itself is therefore the resultant of the system (F, F_1 , F_2) as well as of force F and torque M. It follows from the properties of a couple that magnitudes of all forces F, F_1 , F_2 are equal and both F and F_2 are opposed to F_1. As by the previous method, the line of action of F_2 can be obtained by a perpendicular shift $|M|/|F_2|=|M|/|F|$. There also exists a thumb rule to find the proper direction of that shift as follows: let the tail of the semicircular arrow of the torque be fitted to the tail of the given force F: the tip of the torque arrow turns towards the side of the resultant.



Exercise 7

Determine the resultant of the force $F=31\,\mathrm{kN}$ and torque $M=14\,\mathrm{kNm}$ given in the figure.



Solution

Any resolution equations written in order to get the resultant would contain components of *F* and *R* only (torque never appears in a force resolution equation). Keeping this in mind, what is the sense and magnitude of the resultant force?

$$R=\ldots\ldots\ldots\mathrm{kN}(\quad)$$

Let the intersection of the resultant and the axis *x* be denoted by x_R . The moment of the resultant equals that of the force *F* and the torque about any point of the plane. Let it be written about the origin:

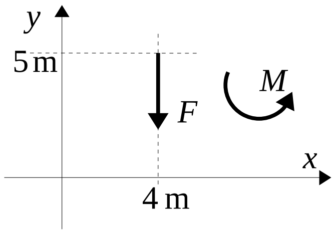
$$\sum M_i^{(0)}: \qquad\qquad \rightarrow x_R=$$

Another option is to find the position of the resultant with respect to force *F*: on which side of the line of action of *F* should the resultant be located in order that it can rotate about the point of application of *F* in the same sense as torque *M* does? ....

How far the resultant should be from force *F* such that it can rotate in the same amount as *M* does?

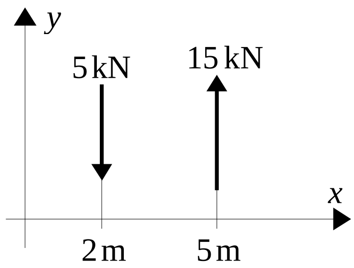
$$\ldots\ldots\ldots=\ldots\ldots\ldots\cdot\Delta x\rightarrow\Delta x=$$

Sketch the result.



Example 8

Find a single force that replaces both forces given in the figure.

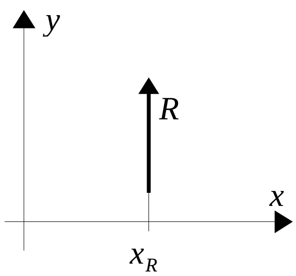


Solution

Assuming a resultant directed vertically upwards, the vertical resolution equation reads:

$$\sum F_{iy}:-5+15=R_y\rightarrow R_y=+10\,\mathrm{kN}(\uparrow)$$

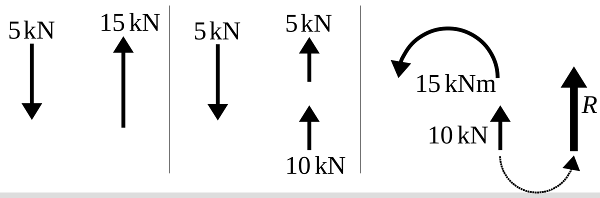
None of the given forces has horizontal components, hence nor does the resultant do: the resultant is therefore a force of 10kN directed upwards vertically. Let the intersection of that resultant and the axis *x* be denoted by x_R -rel. Let the balance of moments about the origin be written as follows:



$$\sum M_i^{(0)}:-5\cdot2+15\cdot5=10\cdot x_R\rightarrow x_R=6.5\,\mathrm{m}$$

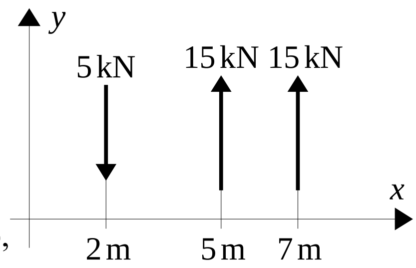
Quick solution using a couple

Let the 15-kN force be split into two components such that one component forms a couple with the 5-kN force and another component will simply the remaining part. The couple has a moment of $+5\cdot3=+15\,\mathrm{kNm}(\curvearrowleft)$, the remaining part is a vertical upwards force of 10 kN in its original line of action. The resultant of that 10-kN force and the 15-kNm torque is another upwards force of 10 kN at a distance of 15/10 = 1.5 m from the original 10-kN (or, 15-kN) force. The sense of that shift of 1.5 m can be found by fitting a counterclockwise semicircular arrow to the tail of the 10-kN force. The tip of the curvilinear arrow will show the side of the force where the resultant is located. Finally, adding those 1.5 metres to the original coordinate of 5 m of the force, the coordinate of the resultant is obtained again as 6.5 m.



Exercise 8

Find a single force that replaces all three forces given in the figure.



Solution

Assuming that the resultant *is* a force, find its components.

All given forces are vertical (with no horizontal components), making the horizontal component of the resultant to be ....

The vertical resolution of the resultant must equal the sum of vetical components of the three forces. Assuming an upwards R_y , the equation is:

$$\sum F_{iy}: F_1 + F_2 + F_3 = R_y \rightarrow \quad ...... \quad ...... \quad ...... = \quad .... \rightarrow R_y = ...\,\mathrm{kN}(\quad)$$

(Since it is nonzero, the resultant is a *force* indeed.)

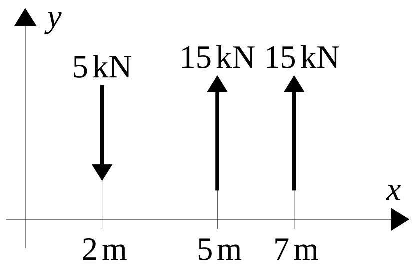
The moment of the resultant force about the origin equals the sum of moments of forces about the origin. Write this equation with the assumption that the resultant intersects axis *x* at x_R :

$$\sum M_i^{(0)}: \quad .....\cdot.... \quad .....\cdot.... \quad .....\cdot.... = \quad .....\cdot....$$

It solves to

$$x_R = ...\,\mathrm{m}$$

What does the sign of x_R mean? Sketch the result.



Resultant of a general plane force system

If a system of forces is not composed exclusively of concurrent forces or exclusively of parallel forces then the force system is called general. The resultant of such a general system can be of any kind discussed so far. For this reason, the complete calculation of the resultant is divided into two steps. Firstly, the type of the resultant is identified, it can then be followed by the numeric evaluation of the resultant.

Identifying the type of resultant: reduction into a force-couple system about a given point

In the first step, let the force system be replaced by a force passing through a point chosen arbitrarily and a moment associated to it (therefore, is also assiciated to the point itself). The names for that force and moment are *net force* and *net couple about* the given point. This system is also called the *equivalent force-couple system* of the original one, which causes the resultant to be the same for both systems. The unknown net couple does not appear in resolution equations, so these equations can be used to get all components of the net force; likewise, the net force passes through the chosen point and has therefore no moment about that, letting the net couple be obtained directly from the equivalence of moments about the chosen point.

If the net force is the zero force and the net couple is also zero, then the resultant is the zero force.

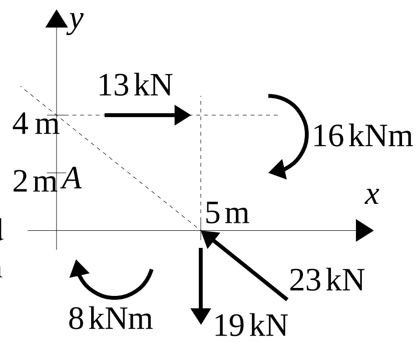
If the net force is the zero force but the net couple is nonzero, then the resultant is a torque with a magnitude equal to that of the net couple.

If the net force is nonzero, then the resultant equals the resultant of the net force and net couple.As was explained earlier, it is always a force with the magnitude and direction of the net force, and whose line of action can be obtained by an appropriate perpendicular translation.

Remark: 'reduction of a system into a force-couple system about point *P*' or 'finding the equivalent force-couple system about point *P*' can also be (in a way somewhat sloppy but with no risk of confusion) referred as 'reduction of the system to point *P*'.

Example 9

Determine the resultant of forces and torques given in the figure by finding their equivalent force-couple system about point A.



Solution

Components of the net force through point *A* can be calculated from simple resolutions (unknown components are assumed in the positive sense of coordinate axes):

$$\sum F_{ix}:+13+0-23\cdot\frac{5}{\sqrt{5^2+4^2}}=A_x \rightarrow \quad \boxed{A_x=-4.960\,\mathrm{kN}(\leftarrow)}$$

$$\sum F_{iy}:0-19+23\cdot\frac{4}{\sqrt{5^2+4^2}}=A_y \rightarrow \quad \boxed{A_y=-4.632\,\mathrm{kN}(\downarrow)}$$

Even at this stage it is clearly seen that the resultant will be a force, whose components are equal to those of force *A* komponenseivel, its magnitude is: $R=\sqrt{4{,}960^2+4.632^2}=6.787\,\mathrm{kN}(\swarrow)$, its direction is given by $\tan\alpha_R=\dfrac{4.632}{4.960}\rightarrow\alpha_R=43.04°$.

The net couple could be in fact determined from a moment about any point but, in order to maintain the calculation intact of any mistake made at the evaluation of force *A*, the most efficient choice is to write the moment about the same point *A*. The oblique force is resolved into components to advantage in its point of intersection made with axis *y*, so it is sufficient to take the horizontal component into account. The net couple is assumed to rotate in a positive sense, thus:

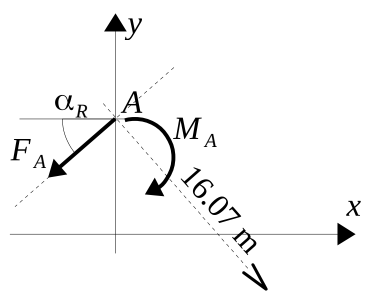
$$\sum M_i^{(A)}:-13\cdot 2-19\cdot 5+23\cdot\frac{5}{\sqrt{5^2+4^2}}\cdot 2-8-16=M_A \rightarrow \quad \boxed{M_A=-109.1\,\mathrm{kNm}(\curvearrowright)}$$

Before the unification of the net force and net couple about point *A* into a final resultant, they are sketched to show the sense of translation of force *A* by the use of the semicircular arrow: it can be concluded that force A should be moved downwards and to the right by an amount of

$\frac{109.1}{6.787}=16.07\,\text{m}$. Horizontal and vertical components of this shift from point A are as follows:

$\Delta x=16.07\cdot\sin 43.04°=10.97\,\text{m}(\rightarrow)$,

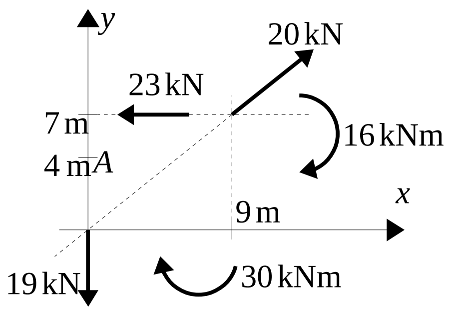
$\Delta y=16.07\cdot\cos 43.04°=11.75\,\text{m}(\downarrow)$.



The line of action passes through the point $\begin{bmatrix}+10.97\\-9.750\end{bmatrix}$, the magnitude of the resultant is $R=6.787\,\text{kN}(\swarrow)$ and it makes an angle $\alpha_R=43.04°$ with the horizontal axis.

Exercise 9

Determine the resultant of forces and torques given in the figure by finding their equivalent force-couple system about point A.



Solution

Among components of the equaivalent force-couple system about A, the net force is calculated first. Let the forces be written into the resolution equations:

$$\sum F_{ix}: \qquad = \quad A_x$$

$$\sum F_{iy}: \qquad = \quad A_y$$

The solution to the equations:

$$A_x=\text{.........}\,\text{kN}(\quad), A_y=\text{.........}\,\text{kN}(\quad)$$

Can the type of resultant be identified on this basis? If yes, what is that? ....

The net couple equals the moment of all forces about point A. (Let us start also here by the forces, writing the moment of the oblique force by components. Force A on the right hand side does not ratate about point A, that is why it is not written out.)

$$\sum M_i^{(A)}: \qquad = \quad \text{......}$$

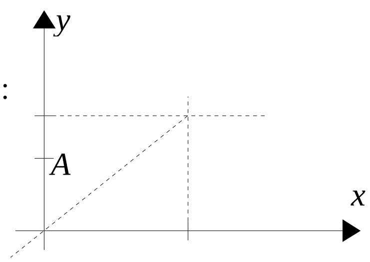
Its solution is:

$$M_A=\text{......}\,\text{kNm}(\text{...})$$

Complete the sketch by the net force and net couple about point A.

What will be the resultant of these two effects? Give its components:

$$R_x=\text{.........}\,\text{kN}(\quad), R_y=\text{.........}\,\text{kN}(\quad)$$



Find the magnitude and direction of the force as well.

$$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ \mathrm{kN}(\ldots),$$

$$\ldots\ldots\alpha_R = \frac{}{} \rightarrow \alpha_R = \ldots\ldots\ {}^\circ$$

Where will be the resultant force with respect to the line of action of force *A*?

What is the amount of translation?

$$\ldots\ldots = \frac{}{} = \ldots\ldots\ldots$$

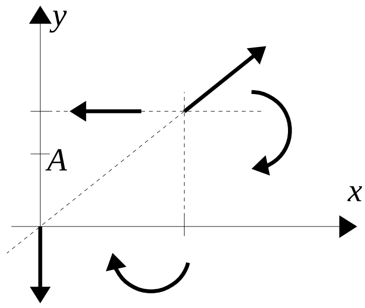
Where this distance should be measured?

Using the porition of point *A*, let us calculate two coordinates of a point on the line of action of the resultant:

$$x_R = 0 \quad \ldots\ldots\cdot\ldots\ldots\ldots \rightarrow x_R =$$

$$y_R = 4 \quad \ldots\ldots\cdot\ldots\ldots\ldots \rightarrow y_R =$$

Make a final sketch.



Distributed loads

The discussion so far has been restricted to the treatment of point loads acting on a material particle. Forces in reality, however, are almost never concentrated to points but are *distributed* in some sense. Distributed forces can be classified according to the domain over which the force is distributed. *Volume forces* or *body forces* are characterized by the property that their resultant depends on the position and magnitude of a three-dimensional region they act upon. A classical representative of such volume forces is gravity that results in a (weight) force of magnitude $\varrho V g$ for each volume V (with ϱ being the density). *Surface forces* are typically contact forces between adjacent bodies. While solving problems in a plane, the third dimension of the problem is projected onto the plane, which results in the transformation of surface loads into *line loads* (or knife-edge loads). The total magnitude of a distributed load obviously depends on the magnitude of the domain acted upon by the load (larger domain typically results in larger resultant) but the ratio of the resultant of the load and the magnitude of the domain has a limit if this latter approaches zero. This value is called the *intensity* of the distributed force at the given point. Distributed forces are commonly denoted by lowercase letters, the most frequently used ones are *g, p, q, w.*

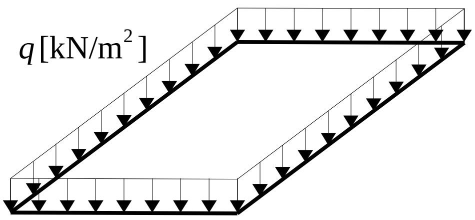
The main objective of this exercise is to learn how distributed loads can be replaced by a single concentrated force, that is, how their resultant can be found.

Resultant of volume forces

If the weight of a straight beam with a rectangular cross section should be characterized by a single force, one can do that by taking the product of mass *m* and acceleration *g* of gravity, where the mass is obtained as the product of volume V and density ϱ: $G=m\cdot g=V\cdot\varrho\cdot g$. The product of density and acceleration of gravity is called specific weight ($\gamma=\varrho\cdot g$, meassured, e.g., in $\mathrm{N/m^3}$), which also corresponds to the intensity of gravity force. The resultant passes the mass centre of the body.

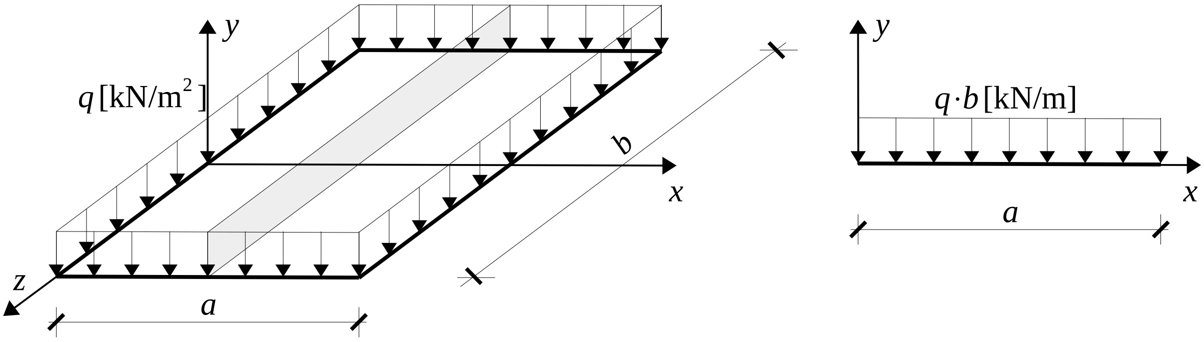
Resultant of surface forces

Intensity of a surface load can be defined in a way quite similar to that of a volume load, just the area of distribution should approach zero in limit. A possible unit for such an intensity is therefore $\mathrm{N/m^2}$. The intensity of a load can be displayed over the surface by perpendicular lines of length proportional to the local value of intensity (perpendicular line lengths in the diagram therefore do not represent real distances but intensity values). The resultant of a surface force is proportional to the volume of this load diagram and passes through its centroid.



Resultant of line forces

Volume or surface loads are often reduced to line loads in practical calculations. If the volume of the beam discussed above is calculated as $V=l\cdot A$ (where l and A denotes the length and cross sectional area of the beam, respectively), then concentration of the volume load in each cross section results in a line load. Its intensity will equal $A\cdot\gamma$ and will then be measured, e.g., in N/m. In a planar problem similar concentration is done for loads distributed over surfaces that are perpendicular to the plane as shown in the following figure:

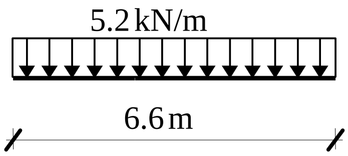


Parallel distributed forces

The resultant is proportional to the 'area' determined by the load diagram, its direction is parallel to the intensity and its line of action passes through the centroid of the diagram. There will be dealt with uniform(ly) and linear(ly) distributed forces, more complicated cases are not discussed here because of the relative complexity of its mathematical background (nevertheless, any properties mentioned so far still extend to those cases as well). Assume that the location of the centroid of two elementary plane shapes (triangle and rectangle) is publicly known, so the resultant can easily be found in elementary cases.

Example 1

Find the resultant of the uniform distributed load shown in the figure.



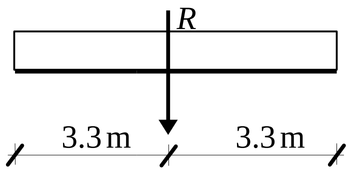
Solution

The resultant of a vertical load is also vertical, its magnitude is:

$$R = 5.2 \cdot 6.6 = 34.32\,\mathrm{kN}\,(\downarrow)$$

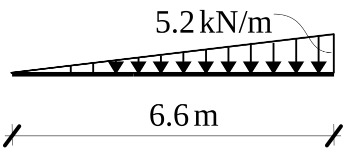
The resultant is located in the middle of the line of distribution:

$$6.6/2 = 3.3\,\mathrm{m}$$



Exercise 1

Find the resultant of the linearly distributed load shown in the figure.



Solution

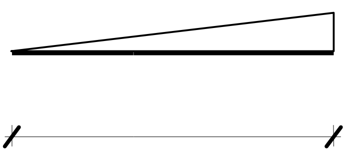
The resultant of a vertical load is also vertical, its magnitude equals the area of the diagram:

$$R=$$

The resultant is located in the third-point of the line of distribution, farther from the zero value:

$$\frac{1}{3}\cdot 6.6 = \qquad \mathrm{m}$$

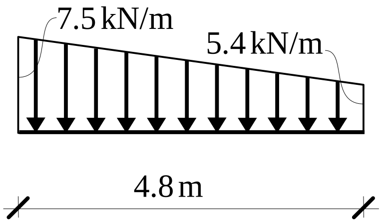
$$\frac{2}{3}\cdot 6.6 = \qquad \mathrm{m}$$



If the load diagram is more complicated than an elementary shape, then its resultant can be obtained by *superposition*. It is based on the fact that the effect of a distributed load is equivalent to the cumulative effect of distributed loads if the cumulative intensity of these equals to the original intensity at each point. In other words, the resultant of a load is obtained as a resultant of partial resultants.

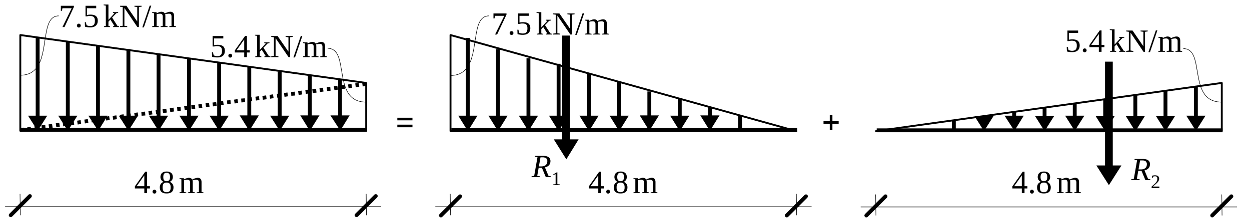
Example 2

Find the resultant of the distributed load shown in the figure.



Solution

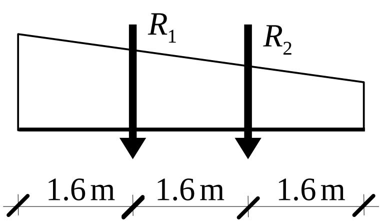
Firstly, the linearly distributed load needs to be replaced by some others of known resultant in a way that component intensities should add up to the original value at each point. It can be done for linear functions if component functions are also (at most) linear and their sum equals the value of the original function at two points (preferably the starting and endpoint). Let both linear functions be chosen such that one of their starting or final values is zero. It implies that values on the opposite end must match the original value of the function.



Two partial resultants are as follows:

$$R_1=\frac{1}{2}7.5\cdot4.8=18\,\text{kN}(\downarrow)$$

$$R_2=\frac{1}{2}5.4\cdot4.8=12.96\,\text{kN}(\downarrow)$$



Their resultant can be obtained from a resolution in *y*:

$$R=R_1+R_2=18+12.96=30.96\,\text{kN}(\downarrow)$$

(This is nonzero, that is why the resultant should have a force component. Otherwise the resultant could either be a couple or equilibrium, both needing a slightly different analysis.)

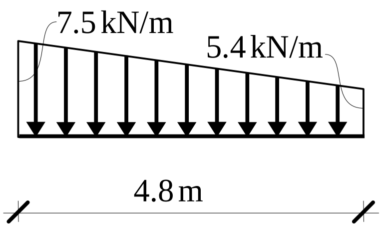
In order to locate the resultant, assume that it stands right to the left endpoint at a distance x_R then write the moments about the same endpoint as follows:

$$\sum M_i^{(l)}:-30.96\cdot x_R=-18\cdot1.6-12.96\cdot3.2$$

that yields $x_R=2.270\text{ m}$.

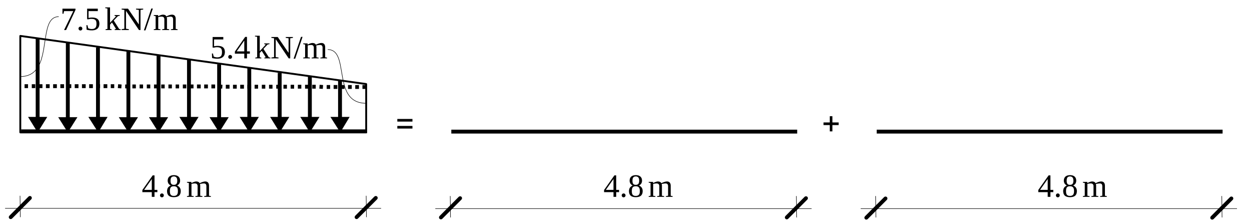
Exercise 2

Find the resultant of the distributed load shown in the figure.



Solution

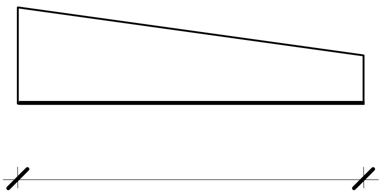
Let the distributed load be split into a uniform and a linear part with an intensity ranging from zero to its maximum:



Calculate partial resultants and mark their position in the figure:

$R_1 =$

$R_2 =$



Summing up (signed) partial resultants:

$$R = R_1 + R_2 =$$

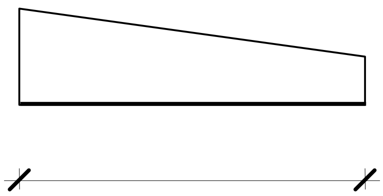
Moment of the resultant must equal the sum of all moments of partial resultants about any points of the plane. Location of the resultant is obtained therefore from a moment equation:

$\sum M_i^{(\ )}:$

This implies the position of the resultant to be

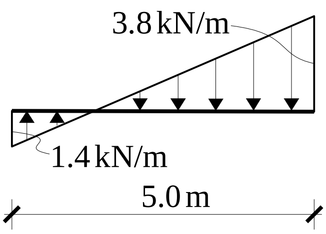
$$x_R =$$

Sketch the resultant in the figure.



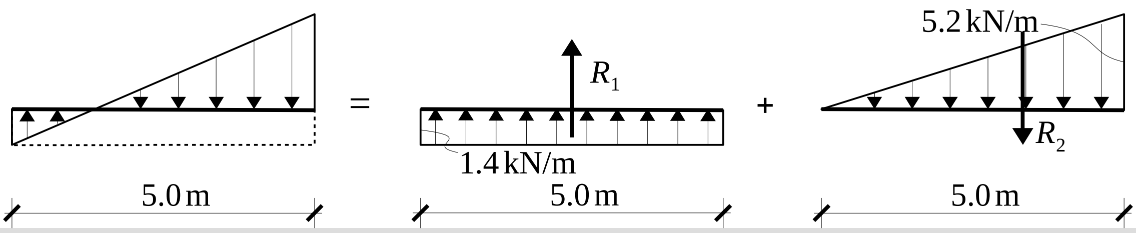
Example 3

Find the resultant of the distributed load shown in the figure.



Solution by a uniform and a linear load

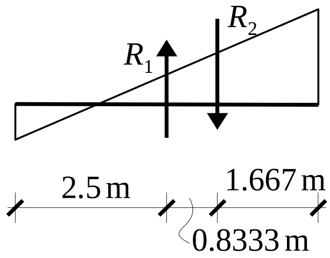
Let the intensity of the uniform load be equal to the starting intensity. The linear load component has to have therefore an intensity at the other end that is still missing from the original intensity: $-1.4 + q_l = 3.8 \rightarrow q_l = 5.2\,\text{kN/m}$. Two component loads and their resultants are now as follows:



Partial resultants:

$$R_1 = -1{,}4 \cdot 5{,}0 = -7{,}0\,\mathrm{kN}\,(\uparrow)$$

$$R_2 = \frac{1}{2}\,5{,}2 \cdot 5{,}0 = 13\,\mathrm{kN}\,(\downarrow)$$



The magnitude of resultant is obtained from resolution:

$$R = R_1 + R_2 = -7.0 + 13.0 = +6.0\,\mathrm{kN}\,(\downarrow)$$

(This is nonzero, that is why the resultant should have a force component. Otherwise the resultant could either be a couple or equilibrium, both needing a slightly different analysis.)

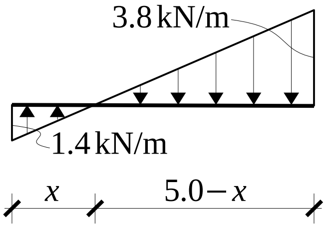
In order to locate the resultant, assume that it stands right to the left endpoint at a distance x_R then write the moments about the same endpoint as follows:

$$\sum M_i^{(l)} : -6.0\,x_R = 7.0 \cdot 2.5 - 13.0 \cdot 3.333$$

that yields $x_R = 4.305\,\mathrm{m}$.

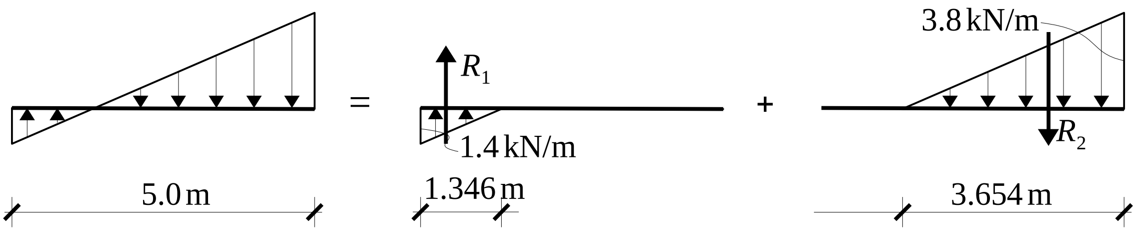
Solution by two non-overlapping linear loads

One might give way to the temptation of calculating the resultant from two triangles that can originally be shown in the figure. For that purpose, the position of zero of the load function should be obtained first, preferably with reference to two similar triangles:



$$\frac{1.4}{x} = \frac{3.8}{5.0 - x} \quad \rightarrow \quad x = 1.346\,\mathrm{m}, \quad 5.0 - x = 3.654\,\mathrm{m}$$

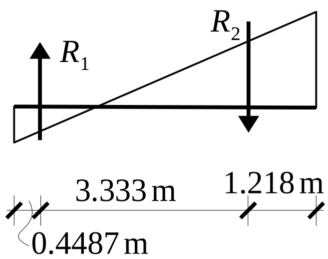
Component loads and their resultants are now as follows:



Partial resultants:

$$R_1 = -\frac{1}{2}\,1.4 \cdot 1.346 = -0.9422\,\mathrm{kN}\,(\uparrow)$$

$$R_2 = \frac{1}{2}\,3.8 \cdot 3.654 = 6.943\,\mathrm{kN}\,(\downarrow)$$



The magnitude of resultant is obtained from resolution:

$$R = R_1 + R_2 = -0.9422 + 6.943 = +6.001\,\mathrm{kN}\,(\downarrow)$$

(This is nonzero, so the resultant will still be determined as earlier...)

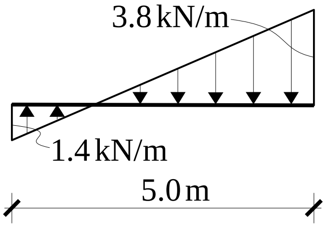
The distance x_R from the left is obtained from moments about the same endpoint as follows:

$$\sum M_i^{(l)} : -6.001\,x_R = 0.9422 \cdot 0.4487 - 6.943 \cdot 3.782$$

that yields $x_R = 4{,}305\,\mathrm{m}$.

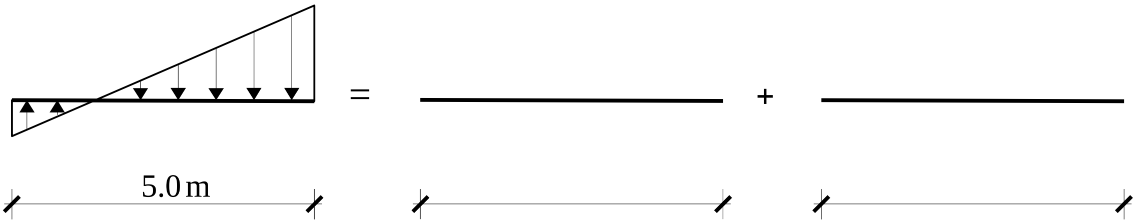
Exercise 3

Find the resultant of the distributed load shown in the figure.

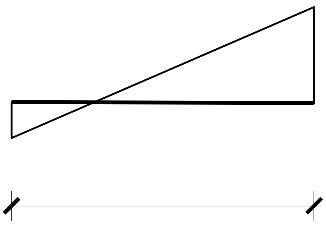


Solution

Let the distributed load be split into two linear parts distributed over the entire length such that their intensities are zero and maximum at the two endpoints. Draw the component loads and the position of their (partial) resultants into the figure:



Calculate partial resultants and mark their position in the figure:



$$R_1 =$$

$$R_2 =$$

Summing up (signed) partial resultants:

$$R = R_1 + R_2 =$$

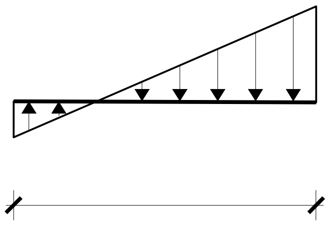
Moment of the resultant must equal the sum of all moments of partial resultants about any points of the plane. Location of the resultant is obtained therefore from a moment equation:

$$\sum M_i^{(\ )}:$$

This implies the position of the resultant to be

$$x_R =$$

Sketch the resultant in the figure.

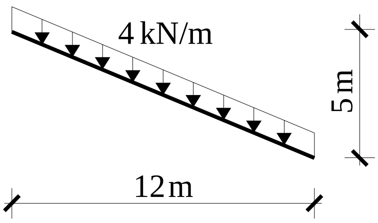


Different kinds of distribution of loads

Linear distributed loads can further be classified according to the measure over which the distribution is defined. Loads distributed *over (member) length* are, e.g., gravity and wind load where the resultant is obtained as a product of the intensity and the length of the loaded member (all examples shown so far shared this property). Another kind of distributed loads are *projected*, e.g., the load of snow. In such cases, the length of distribution is understood as a projection of the member length and the resultant is obtained by a multiplication with that projected length.

Example 4

a) Find the resultant of the uniform load distributed over the length of the beam as shown in the figure.



Solution

The length of distribution,

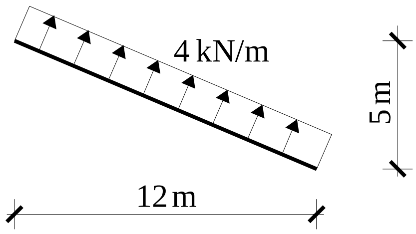
$$l=\sqrt{5^2+12^2}=13.00\,\mathrm{m}$$

Magnitude of the resultant is obtained by multiplication:

$$R=4\cdot 13.00=52\,\mathrm{kN}$$

The resultant bisects the load diagram vertically and is directed downwards.

b) Find the resultant of the uniform load distributed over the length of the beam as shown in the figure.



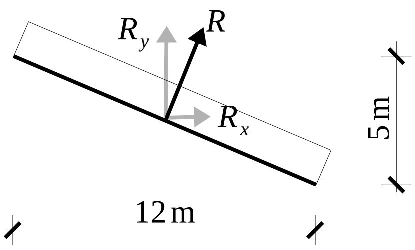
Solution

The length of distribution,

$$l=\sqrt{5^2+12^2}=13.00\,\mathrm{m}$$

Magnitude of the resultant is obtained by multiplication:

$$R=4\cdot 13.00=52\,\mathrm{kN}$$

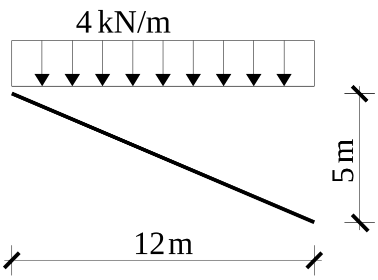


This oblique force is perpendicular to the line of distribution and makes the same angle with the vertical direction as the line of the load makes with the horizontal. Its components are as follows:

$$R_x=52\cdot\frac{5}{\sqrt{5^2+12^2}}=20\,\mathrm{kN}(\rightarrow)$$

$$R_y=52\cdot\frac{12}{\sqrt{5^2+12^2}}=48\,\mathrm{kN}(\uparrow)$$

c) Find the resultant of the uniform projected load as shown in the figure.



Solution

Now the length of distribution is the horizontal projection:

$$l=12\,\mathrm{m}$$

Magnitude of the resultant is obtained by multiplication:

$$R=4\cdot 12.00=48\,\mathrm{kN}$$

The resultant bisects the load diagram vertically and is directed downwards.

Exercise 4

a) Find the resultant of the uniform load distributed over length as shown in the figure.

Solution

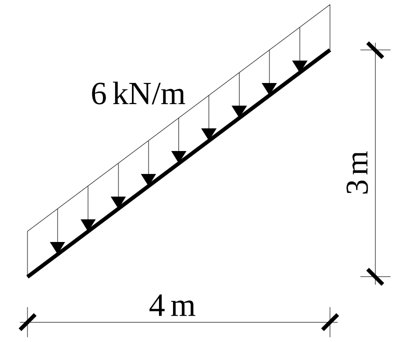
The length of distribution,

$$l=$$

Magnitude of the resultant is obtained by multiplication:

$$R=$$

Location and direction of the resultant:..



b) Find the resultant of the uniform load distributed over length as shown in the figure.

Solution

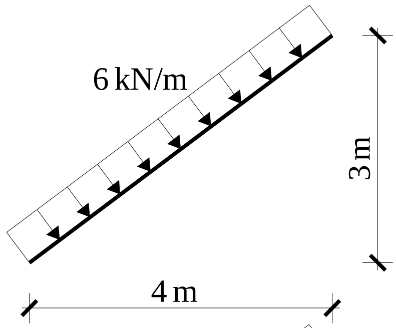
The length of distribution,

$$l=$$

Magnitude of the resultant is obtained by multiplication:

$$R=$$

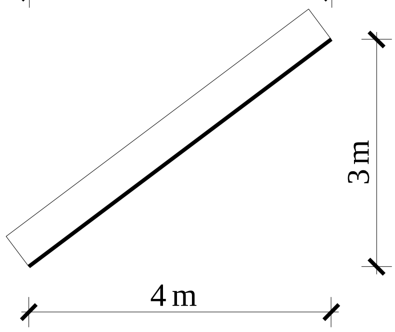
Location and direction of the resultant:..



Horizontal and vertical components of R:

$$R_x=$$

$$R_y=$$



c) Find the resultant of the uniform projected load as shown in the figure.

Solution

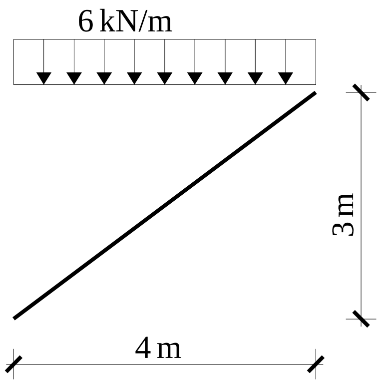
The (horizontal) length of distribution,

$$l=$$

Magnitude of the resultant is obtained by multiplication:

$$R=$$

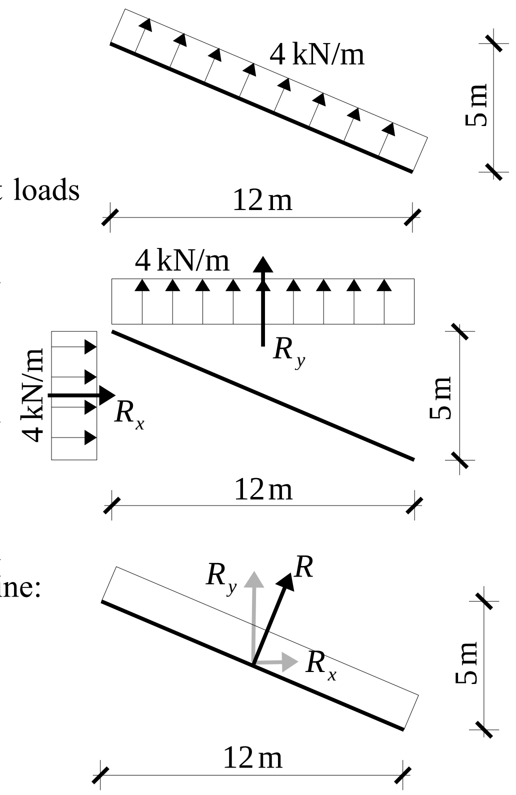
Location and direction of the resultant:..



Resultants of loads that act perpendicularly to their line of distribution can also be obtained in an alternative way (in addition to wind load, water pressure also belongs to this kind of loads). After a careful consideration of the problem one can conclude that vertical and horizontal components of an oblique force can even be dealt with as distributed loads independent of each other.

Example 5

Find the resultant of the uniform load distributed over length as shown in the figure.



Solution

A perpendicular load can be replaced by two component loads distributed over horizontal and vertical projected lengths.

The resultant of the horizontal load distributed over a height of 5 metres has a rightwards resultant:

$$R_x = 4 \cdot 5 = 20\,\mathrm{kN}(\rightarrow)$$

The resultant of the vertical load distributed over a width of 12 metres has an upwards resultant:

$$R_y = 4 \cdot 12 = 48\,\mathrm{kN}(\uparrow)$$

Both resultants bisect the corresponding load diagram and they add to an oblique resultant in the middle of the line:

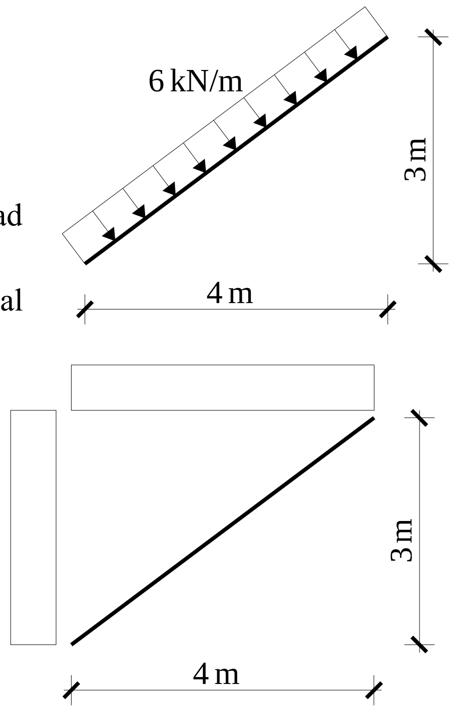
$$R = \sqrt{20^2 + 48^2} = 52.00\,\mathrm{kN}(\nearrow)$$

Its inclination to the horizontal is

$$\alpha_R = \arctan\frac{48}{20} = 67.38\,°$$

Exercise 5

Find the resultant of the uniform load distributed over length as shown in the figure.



Solution

Replace the load acting on the line by two projected load components.

Find the resultants of both horizontal and vertical components:

$R_x =$

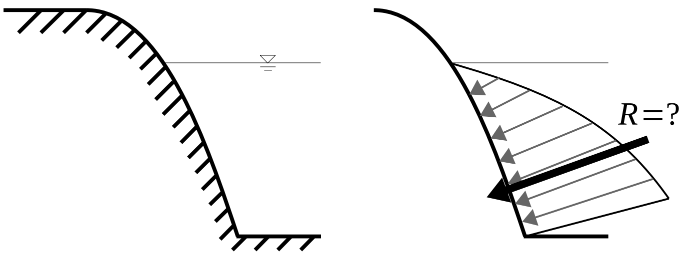
$R_y =$

Draw them into the figure and calculate the resultant of two components:

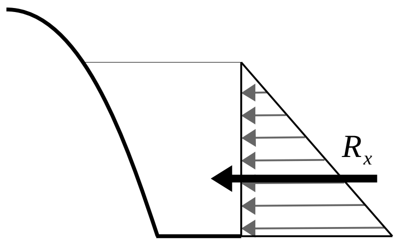
$R =$

$\alpha_R =$

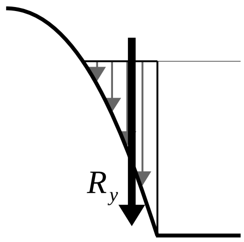
Generally speaking it is easier to perform calculations with horizontal and vertical load components, which supports the convention of leaving the resultant in components instead of re-expressing it in terms of magnitude and direction.



Water pressure also acts on surfaces as a distributed load. It is characterized by the property that its intensity is always locally perpendicular to the surface and is proportional to the depth of water. It seems to cause complications in finding the resultant as the problem requires operations on a force of variable intensity and curved line of distribution. In fact, the calculation is surprisingly simple due to the two properties mentioned above.



Horizontal component can be obtained as a resultant of a projected linear load with a triangular diagram: that force must pass the lower third-point of the height.



The intensity of the vertical load component, however, is always proportional to the depth of water. With an appropriate scale chosen, its function of intensity can be demonstrated by the plane shape enclosed by the dam and the water level. If both the area and centroid of that plane shape are known, then the vertical component of the resultant of water pressure can also be found.

Centroid of composite shapes

Coordinates of the *centroid* of a shape embedded in the plane *yz* can be found by the following formulae:

$$y_C = \frac{\int_A y \, \mathrm{d}A}{\int_A \mathrm{d}A} = \frac{Q_z}{A} \quad z_C = \frac{\int_A z \, \mathrm{d}A}{\int_A \mathrm{d}A} = \frac{Q_y}{A} \ ,$$

where A represents the area, and Q_z and Q_y stand for the *first moment* taken about axes z and y. The first moment of a shape about an axis is obtained as a (signed) product of the area of the shape and the distance between the centroid of the shape and the axis; thus, it is measured in, e.g., in m^3.

The centroid of a composite shape can be located by splitting the shape into simpler ones with known centroidal coordinates. This way, both the area and first moment of the shape can be found as the sum of composite areas and first moments, respectively:

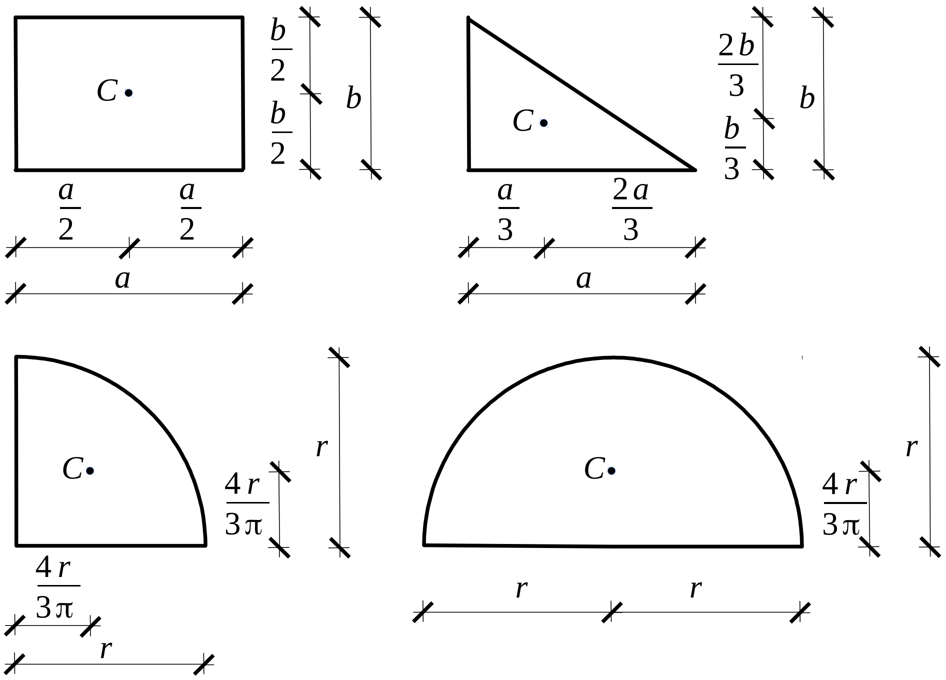
$$A = \sum_i \int_{A_i} \mathrm{d}A = \sum_i A_i \ ,$$

$$Q_z = \sum_i Q_{zi} = \sum_i y_{Ci} A_i \ ,$$

$$Q_y=\sum_i Q_{yi}=\sum_i z_{Ci}A_i \ .$$

Here A_i, Q_{zi}, Q_{yi} denote the area and first moments about axes *z* and *y* of a shape component, while y_{Ci}, z_{Ci} are coordinates of the centroid of each shape component.

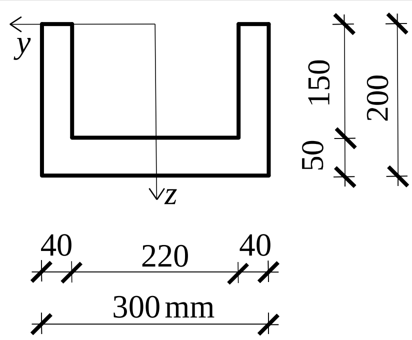
This method can only be applied if the centroid of all (elementary) shape components are known. It is worth memorizing the location of centroid of four elementary shapes shown in the figure below:



Using the technique of components introduced above, it is easy to see that the centroid of a symmetric object must be incident to the axis of symmetry, since first moments of shape components about the axis of symmetry on either side of the same axis are negative of each other, so the resultant first moment becomes zero.

Example 6

Locate the centroid of the shape shown in the figure using the method of decomposition and make a final sketch.



Solution

Since the shape is symmetric, its centroid is incident to the axis of symmetry. For this reason the coordinate system is set such that axis *z* coincides with the axis of symmetry; therefore, only a coordinate *z* of the centroid should be calculated.

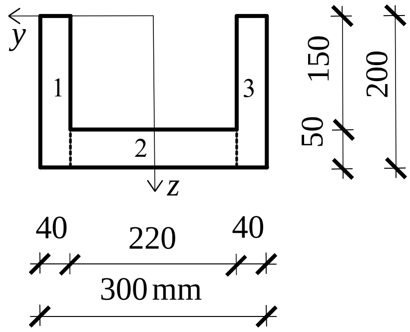
Let the shape be split into three rectangles as shown in the figure.

Total area of the shape equals the sum of area components:

$$A=A_1+A_2+A_3=200\cdot 40+220\cdot 50+200\cdot 40=27000\,\text{mm}^2\ .$$

Total first moment of the shape about axis y can be found as the sum of first moments of three shape components about the same. The first moment of each shape component can be found as the product of its area and distance measured from axis y:

$$Q_y = Q_{y1} + Q_{y2} + Q_{y3} =$$
$$200 \cdot 40 \cdot 100 + 220 \cdot 50 \cdot (150 + 25) + 200 \cdot 40 \cdot 100 = 3525000\,\text{mm}^3$$

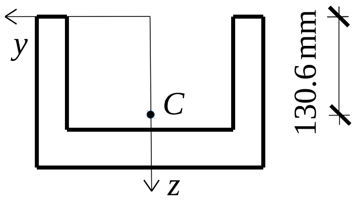


Coordinate z (i.e., distance measured from axis y) of the centroid of the composite shape is then found as the ratio of the total first moment about axis y and the area:

$$z_C = \frac{Q_y}{A} = \frac{3525000}{27000} = 130.6\,\text{mm}$$

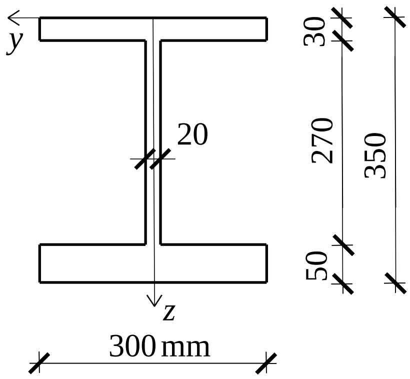
Finally, display the centroid together with the distance z_C in a final sketch.

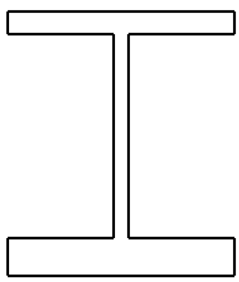
Final sketch:



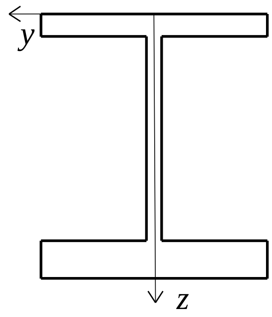
Exercise 6

Locate the centroid of the I section shown in the figure using the method of decomposition and make a final sketch.





Final sketch:



The centroid of a composite shape can also be located by completing the shape by (some) simple shape(s) in a way that the resultant shape is still simple, even if large, and the position of its centroid is known. The resultant area in this case can be obtained as the difference between the completed (total) area and the area of completion:

$$A = A_{\text{total}} - A_{\text{completion}} \ .$$

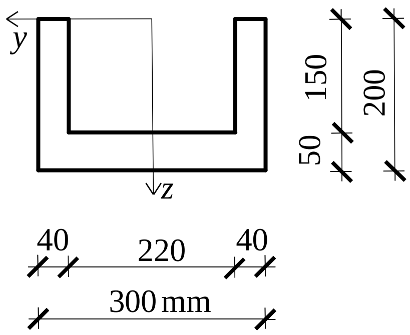
First moments are obtained in a similar manner by subtracting the first moment of the area of completion from that of the total area:

$$Q_z = Q_{z,\text{total}} - Q_{z,\text{completion}} \ ,$$

$$Q_y = Q_{y,\text{total}} - Q_{y,\text{completion}} \ .$$

Example 7

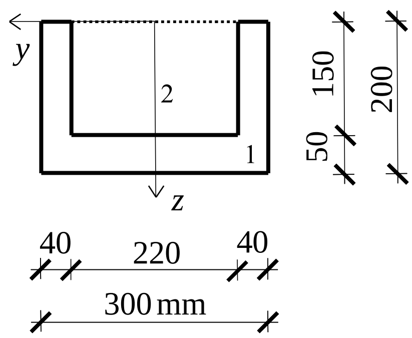
Locate the centroid of the shape shown in the figure using the method of completion and make a final sketch.



Solution

Since the shape is symmetric, its centroid is incident to the axis of symmetry. For this reason the coordinate system is set such that axis z coincides with the axis of symmetry; therefore, only a coordinate z of the centroid should be calculated.

Let the shape be completed to a rectangle as shown in the figure. Total area of the shape equals the difference of the large and small rectangles:



$$A = A_1 - A_2 = 300 \cdot 200 - 220 \cdot 150 = 27000 \,\text{mm}^2 \ .$$

Similarly, total first moment of the entire shape about axis y can be found as the difference of first moments of rectangular shapes about the same:

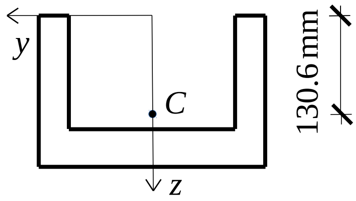
$$Q_y = Q_{y1} - Q_{y2} = 300 \cdot 200 \cdot 100 - 220 \cdot 150 \cdot 75 = 3525000 \,\text{mm}^3 \ .$$

Coordinate z (i.e., distance measured from axis y) of the centroid of the composite shape is then found as the ratio of the total first moment about axis y and the area:

$$z_C = \frac{Q_y}{A} = \frac{3525000}{27000} = 130.6\,\text{mm}$$

Finally, display the centroid together with the distance z_C in a final sketch.

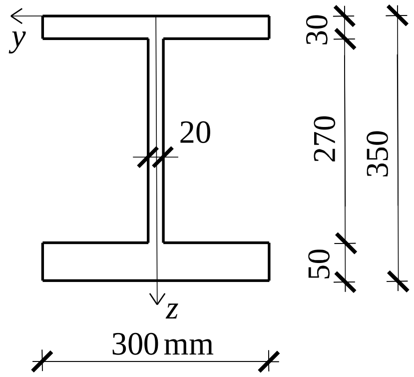
Final sketch:

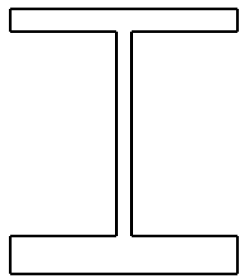


Exercise 7

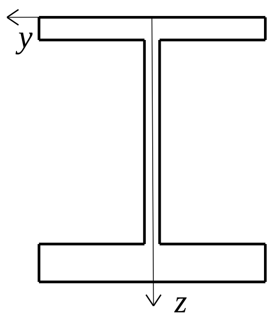
Locate the centroid of the I section shown in the figure using the method of completion and make a final sketch.

Solution



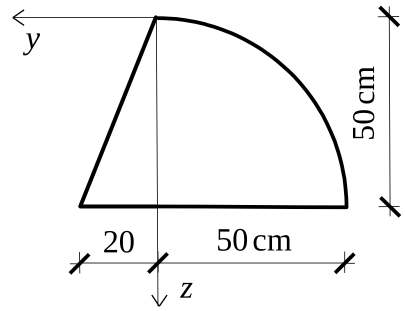


Final sketch:



Example 8

Locate the centroid of the shape shown in the figure.
Make a final sketch.

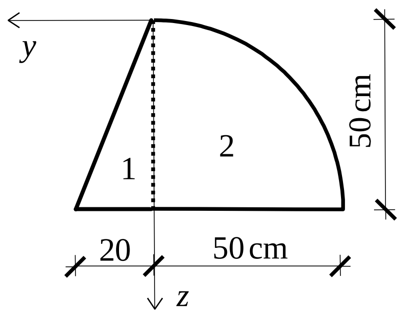


Solution

Let the shape be split into a quarter-circle and a triangle as shown in the second figure.

The total area of the shape equals the sum of area components:

$$A = A_1 + A_2 = \frac{1}{2} \cdot 50 \cdot 20 + \frac{1}{4} \cdot 50^2 \cdot \pi = 500 + 1963 = 2463 \, \text{cm}^2 .$$



Total first moment of the shape about axis y can be found as the sum of first moments of the rectangle and the quarter-circle:

$$Q_y = Q_{y1} + Q_{y2} = 500 \cdot \frac{2}{3} \cdot 50 + 1963 \cdot \left(50 - \frac{4 \cdot 50}{3 \cdot \pi}\right) = 73170 \, \text{cm}^3 .$$

Coordinate z (i.e., distance measured from axis y) of the centroid of the composite shape is then found as the ratio of the total first moment about axis y and the area:

$$z_C = \frac{Q_y}{A} = \frac{73170}{2463} = 29.71 \, \text{cm}$$

In order to get coordinate y of the centroid, let first moments of area components about axis z be calculated:

$$Q_y = Q_{y1} + Q_{y2} = 500 \cdot \frac{1}{3} \cdot 20 - 1963 \cdot \frac{4 \cdot 50}{3 \cdot \pi} = -38330 \, \text{cm}^3 .$$

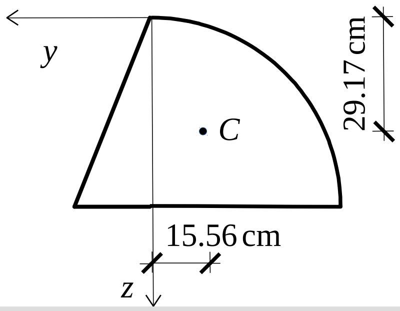
(Warning: irst moment of the quarter-circle is negative since its centroidal coordinate y is also negative.)

Coordinate y (i.e., distance measured from axis z) of the centroid of the composite shape is then found as the ratio of the total first moment about axis z and the area:

$$y_C = \frac{Q_z}{A} = \frac{-38330}{2463} = -15.56 \, \text{cm} .$$

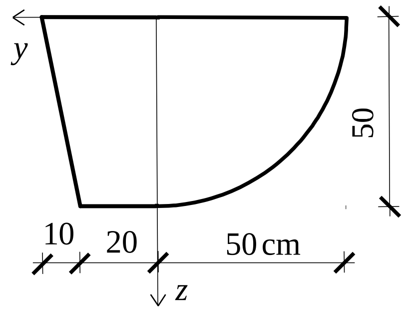
Finally, display the centroid together with the distances y_C and z_C in a final sketch.

Final sketch:



Exercise 8

Locate the centroid of the shape shown in the figure.
Make a final sketch.



Solution

Rigid bodies

Analysis of motions of rigid bodies is based on the assumption that a rigid body is equivalent to a set of connected (elementary) massive particles. Those particles are also in interaction with each other, implying that each particle moves under the effect of forces acting upon it, although still obeying the conditions imposed by the concept of rigidity. Forces due to internal connections, however, cancel each other while summing up (integrating) equations of motion written for individual particles. As a result, rigid-body motion is influenced only by forces that are external to the body.

Kinematics of rigid bodies

As in the case of particles, a precise description of the position of the rigid body should be given first. The discussion will now be restricted to two-dimensional systems only: each point is assumed to move parallel to a plane fixed beforehand. Plane motion of a body can be *translational*, *rotational about a fixed axis* or a *general two-dimensional motion*.

Translation

A motion is called translational if all points of the body undergo the same displacement with respect to their original position. Consequently, the body does not suffer any rotation compared to its original position and all its points share the same velocity and acceleration. In such a case (and only in such a case) it is possible to speak about *the* velocity of a rigid body.

Rotation about a fixed axis

A motion is called rotational if all points of the body rotate about the same fixed axis. In comparison to the original configuration, angular displacement (rotational angle) is common to all points. Consequently, its rates of change in time, i. e., angular velocity and angular acceleration are also common to any points (these values can therefore be regarded as kinematic parameters of *the body*). At the same time, (linear) velocity and acceleration of an individual point depends on the distance between the point and the axis of rotation (recall that $v=\omega\cdot r, a_n=\omega^2\cdot r, a_\tau=\kappa\cdot r$), that is, it can vary from point to point. For this reason, it makes no sense to speak about *the* velocity of a rotating body.

General plane motion

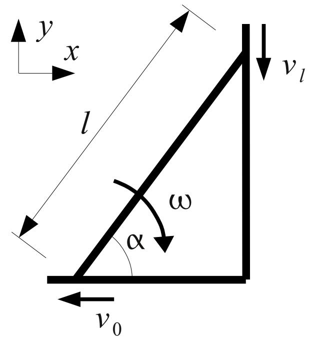
A general plane motion can always be regarded as a combination of a rotational motion about an arbitrarily chosen axis and a translational motion defined by the displacement of the same axis. In doing so, however, the *pivot point* of the body (where the axis of rotation is thought to pass) must always be specified. The translational component is defined by the displacement of the pivot point. Since angular velocity and angular acceleration are both zero for the translational part, they cannot depend on the choice of the pivot point either. Vectors of (linear) velocity and acceleration of an individual point can be formed as a vectorial sum of components calculated from translational as well as rotational parts of the motion.

A pivot point can be chosen freely (it can either be an external point with an imaginary connection to the physical body). In most of cases, however, two points of particular importance are used. One of them is the *centre of mass* of the body, the other is a point whose velocity is just zero in a time instant. This latter point is referred to as *instantaneous centre of rotation* of the body.

Since the velocity of the instantaneous centre is zero, only the effect of rotation should be considered in calculating the velocity of another point (warning: it does not extend automatically to accelerations). Thus, magnitude of the velocity of an arbitrary point depends only on angular velocity and the distance between the point and the instantaneous centre; the direction of velocity is always perpendicular to the radius drawn to the point from the instantaneous centre. It follows from the above facts that, knowing the vectors of velocity in two points within a rigid body, the instantaneous centre can be obtained. Draw a line perpendicular to each vector of velocity through the point it belongs to: those two lines intersect exactly at the instantaneous centre.

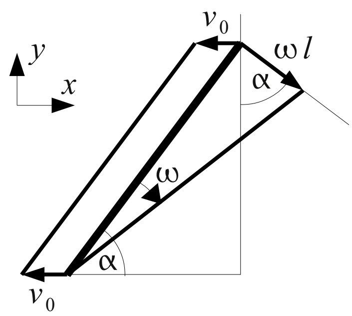
Example 1

A ladder of length $l=4\,\mathrm{m}$ supported against the wall is sliding. downwards. After a while, the ladder makes an angle $\alpha=50°$ with the horizontal. In the same time the bottom of the ladder travels with a velocity $v_0=2\,\mathrm{m/s}$ and acceleration $a_0=0.5\,\mathrm{m/s^2}$ (both directed apart from the wall).
Find the velocity and acceleration of the top of the ladder still contacting the wall at that instant.



Solution

Horizontal and vertical components of velocity of the top of the ladder at the wall can be written separately in terms of parameters of both translational and rotational motions of the bottom point (those components are shown in the figure in the middle). Due to the continuous contact with the wall, horizontal component of the velocity should be zero:



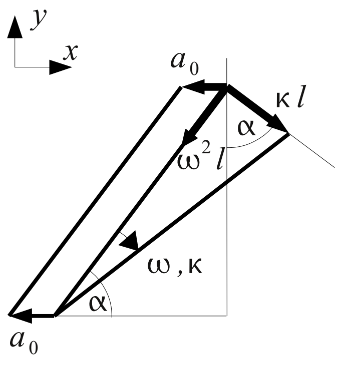
$$v_{lx}=-v_0+\omega\cdot l\cdot\sin\alpha=0\quad ,$$

hence angular velocity of the ladder is $\omega=0.6527\,\mathrm{rad/s}\,(\curvearrowright)$.

The vertical component itself is the velocity in case:

$$v_{ly}=0+\omega\cdot l\cdot\cos\alpha\rightarrow\qquad \boxed{v_{ly}=1.678\,\mathrm{m/s}\,(\downarrow)}$$

With a similar reasoning, acceleration of the top point can be written in terms of parameters of the bottom one by three components shown in the figure at the bottom as well. Due to the continuous contact with the wall, horizontal components of the acceleration should add up to zero:



$$a_{lx}=-a_0+\kappa\cdot l\cdot\sin\alpha-\omega^2\cdot l\cdot\cos\alpha=0\quad ,$$

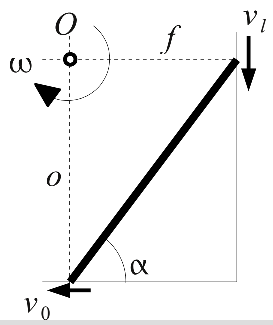
hence angular acceleration of the ladder is $\kappa=0.5206\,\mathrm{rad/s^2}\,(\curvearrowright)$

The vertical component itself is the acceleration in case:

$$a_{ly}=0-\kappa\cdot l\cdot\cos\alpha-\omega^2\cdot l\cdot\sin\alpha\rightarrow\qquad \boxed{a_{ly}=-2.644\,\mathrm{m/s^2}\,(\downarrow)}$$

Finding velocity by the use of the instantaneous centre

The instantaneous centre of a moving rigid body is located at the point (rigidly attached to the body) whose velocity is zero. Velocity pertaining to a point of the body is always perpendicular to the line drawn through the instantaneous centre, and its magnitude is found as the product of the distance from the centre and the angular velocity.

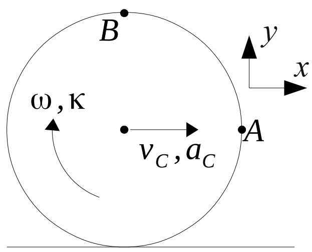


In the current problem, known directions of velocities make sure that the instantaneous centre should be incident to both lines *o* and *f*; yielding formulae

$$v_0=\omega l \sin\alpha, \quad v_l=\omega l \cos\alpha \quad .$$

Thus, $\dfrac{v_0}{\sin\alpha}=\dfrac{v_l}{\cos\alpha} \rightarrow v_l=\dfrac{v_0}{\tan\alpha}$, that coincides with the result obtained earlier.

Exercise 1



Angular velocity, angular acceleration, as well as the velocity and acceleration of the centre of mass of a wheel of radius $R=0.5\,\mathrm{m}$ is given.
Find velocity at the bottom point and at point A, as well as the acceleration at point B.
$\omega=3\,\mathrm{rad/s}, \kappa=1.6\,\mathrm{rad/s^2}, v_C=1.2\,\mathrm{m/s}, a_C=2.2\,\mathrm{m/s^2}$

Solution

Velocities are calculated from the velocity of the centre of mass as a translational part and from the angular velocity ω about the same point as a rotational part. At the bottom point:

$$v_x=$$

$$v_y=$$

At point *A*:

$$v_{Ax}=$$

$$v_{Ay}=$$

The components of acceleration are obtained from the acceleration of the centre of mass as a translational part and from a rotation about the same (with angular velocity ω and angular acceleration κ) as follows:

$$a_{Bx}=$$

$$a_{By}=$$

The translational motion and the rotational motion about a fixed axis represent, of course, only two special cases of a general plane motion. A roataion about any fixed axis can therefore be defined as a motion whose instantaneous centre is always located at the axis of rotation. Conversely, the rotation, angular velocity and angular acceleration are always zero for a translational motion, hence no instantaneous cenre can be defined for that special case.

Kinetics of rigid bodies

Problems in 2D require three scalars to specify the exact position of a rigid body in the plane, so any motion can be described by three independent equations. In order to set up those equations, theorems based on the summation of laws of motion for particles are used.

Euler's first law of motion (an extension of Newton's second law to rigid bodies)

***Theorem**:* The cenre of mass of a rigid body moves exactly in a way as if it was a massive particle, acted upon by the same loads as the rigid body is exposed to and having the same mass as the rigid body has. (Formally, $\boldsymbol{R}=m\cdot\boldsymbol{a}_C$.)

As can be seen clearly, the point whose acceleration is proportional to the net force acting on the body must be specified. Euler's first law can be transformed into scalar equations exactly as it has already been shown for material particles, and there can also be found only two independent equations among infinitely many possibilities.

Euler's second law

A counterpart of the former theorem concerns the rate of change of velocity of the rotational motion. It will be presented as an adaptation of Newton's second law to rotations. According to the vocabulary of linear and rotational kinematic parameters, 'acceleration' should be replaced by 'angular acceleration'. The role of 'mass' gets transferred to the *moment of inertia* about a chosen axis and, instead of considering force resolutions, moments of forces and couples about the same axis should be calculated. Thus, the theorem will assume the general form $M=I\cdot\kappa$ in our two-dimensional problems. The chosen axis usually passes through either the centroid or the instantaneous centre, so the theorem can be written in two typical forms as follows:

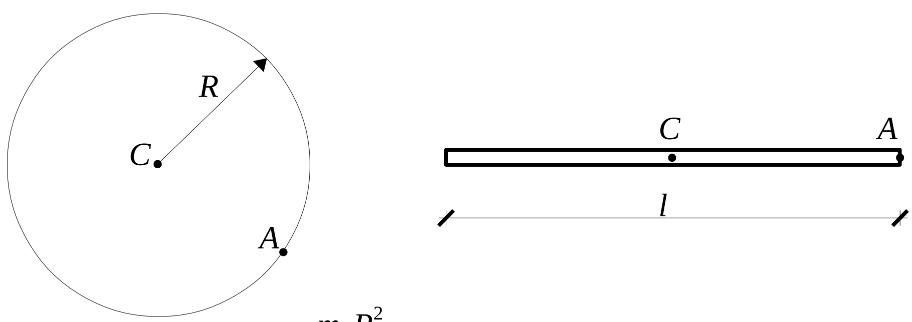
Theorem: The angular acceleration of a rigid body is proportional to the moment of net forces and couples about the centre of mass of the body. The factor of proportionality is then the moment of inertia taken about the centre of mass of the body. (Written as a formula, $M_C=I_C\cdot\kappa$.)

Theorem: The angular acceleration of a rigid body is proportional to the moment of net forces and couples about the instantaneous centre of rotation. The factor of proportionality is then the moment of inertia taken about the instantaneous centre of rotation of the body. (Written as a formula, $M_0=I_0\cdot\kappa$.)

Moment of inertia of a rigid body

The moment of inertia of a rigid body about an axis is defined as a sum of products $r^2\,\mathrm{d}m$ over all elementary massive particles the body is composed of, where d*m* denotes elementary mass and *r* is the distance between the particle and the axis. Written as a formula, $I=\int_m r^2\,\mathrm{d}m$.

Within the scope of this subject, only kinetics of either cylindrical bodies or long, straight and slender rods are dealt with, their moments of inertia are shown in the following figure.



about the centre of mass: $I_C=\dfrac{m\cdot R^2}{2}$ about the centre of mass: $I_C=\dfrac{m\cdot l^2}{12}$

about a point on the perimeter: $I_A=\dfrac{3}{2}m\cdot R^2$ about an endpoint: $I_A=\dfrac{m\cdot l^2}{3}$

Moments of inertia of a cylinder and a straight rod

It is true in general that the moment of inertia of highest importance is that about the center of mass.This has the smallest value and makes possible to obtain moments of inertia about any further axis passing through a point *P* by means of the Parallel axis (or Steiner's) theorem: $I_P = I_C + m \cdot r_{PC}^2$, where r_{PC} denotes the distance between points *P* and *C*.

This means that it should have been sufficient to specify the moment of inertia about the centre of mass, since:

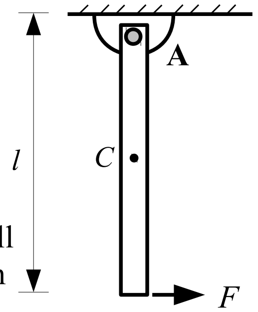
$$I_A = I_C + m \cdot r_{AC}^2 = \frac{1}{2} m \cdot R^2 + m \cdot R^2 = \frac{3}{2} m \cdot R^2$$

$$I_A = I_C + m \cdot r_{AC}^2 = \frac{m \cdot l^2}{12} + m \cdot \left(\frac{l}{2}\right)^2 = \left(\frac{1}{12} + \frac{1}{4}\right) m \cdot l^2 = \frac{m \cdot l^2}{3}$$

When solving a problem, one can choose among resolution equations and two forms of Euler's second law.

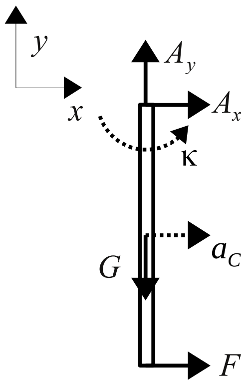
Example 2

Find the angular acceleration of the bar of length $l = 1.2\,\mathrm{m}$ and weight $G = 500\,\mathrm{N}$, as well as the force exerted on the bar by the support A. The bar starts from rest under the action of the force $F = 300\,\mathrm{N}$.



Solution

The bottom figure shows forces acting upon the body in solid lines, as well as angular acceleration and the linear acceleration of the centre of mass in dotted lines. Since the body rotates about a fixed point *A*, these latter two are not independent of each other: $a_C = \kappa \cdot l/2$.



The body starts from rest, so its initial angular velocity equals zero, implying that normal acceleration of the centre of mass is also zero.

The mass of the bar is $m = \frac{500}{9.81} = 50.97\,\mathrm{kg}$.

Euler's first law in *x* and *y* yields that:

$$A_x + 300 = 50.97 \cdot a_C$$

$$A_y - G = 50.97 \cdot 0$$

Due to the fixed axis, Euler's second law can be written in both for the centre of mass and for the centre of rotation:

$$300 \cdot 0.6 - A_x \cdot 0.6 = \frac{50.97 \cdot 1.2^2}{12} \cdot \kappa$$

$$300 \cdot 1.2 = \frac{50.97 \cdot 1.2^2}{3} \cdot \kappa$$

Clearly, only three out of these four equations are needed for the solution. From the last one,

$$\boxed{\kappa = 14.71\,\mathrm{rad/s^2} (\curvearrowleft)}$$

Plugging this back into the first and second equations we have

$$\boxed{A_x = 150.0\,\mathrm{N} (\rightarrow), A_y = 500.0\,\mathrm{N} (\uparrow)}$$.

Exercise 2

A beam of mass $m=15\,\mathrm{kg}$ and length $l=1.5\,\mathrm{m}$ is held at both ends.
Find the angular aceleration of the beam if one end is suddenly released.
Find the force needed at the other end at the same instant in order to prevent it from moving.

Solution

Draw forces acting on the beam on release. Mark the expected rotation, angular acceleration as well as the acceleration of the centre of mass.

Write two equations based on Euler's first law:

$$\sum F_{ix}:$$

$$\sum F_{iy}:$$

Calculate moments of inertia about axes passing through the centre of mass and of rotation:

$I_C=$ $\qquad\qquad\qquad$ $I_0=$

Write Euler's second law about the centre of mass and centre of rotation:

$$\sum M_i^{(C)}:$$

$$\sum M_i^{(0)}:$$

Solve the system of equations:

$\kappa=$

$A_x=$

$A_y=$

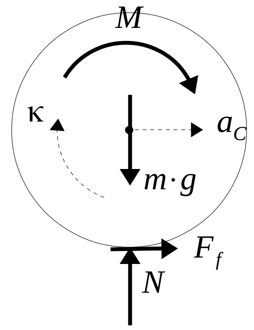
Rolling wheel

The motion of a wheel rolling on a plane, horizontal or inclined, can be classified according to whether or not its contact point has a relative velocity with respect to the plane. If yes, then the we speak about *rolling with slipping*: in that case, friction is termed *kinetic friction* and acceleration of the centre of mass is independent of angular acceleration. If velocities of the plane and the contact point of the wheel coincide, the motion is called *rolling without slipping* or *pure rolling*. A wheel rolling without slipping has always its instantaneous cenre of rotation located at the point of contact. At the same time, velocity of the centre of mass is related to angular velocity as $v_C=\omega\cdot R$, as well as acceleration of the centre of mass is similarly related to angular acceleration ($a_C=\kappa\cdot R$). In the case of pure rolling, the force of friction, termed *static friction* is independent of the tightening force or normal force between contacting parts (that is why there is no compulsory sense of its assumed direction: at most a negative result means its reversion in the final form), but the law $|F_f|\leq\mu\cdot N$ of static friction can only be satisfied with sufficiently large values of the coefficient of friction; that inequality must therefore be checked.

Example 3

A wheel of mass $m = 20\,\text{kg}$ and radius $R = 0.30$ m is driven by a torque of $M = 50\,\text{Nm}$ on a horizontal plane.
Find the acceleration of the centre of mass if the wheel rolls without slipping.
Find minimum coefficient of friction required for a pure rolling.

Solution



The figure to the right shows the driving torque in a clockwise sense and all other forces exerted upon the wheel. The torque would result in a leftwards slipping in the lack of friction, so the force of static friction is rather assumed to point to the right, though 'wrong' assumption of a *static* force component would not influence the final result. Dotted lines denote expected senses of angular acceleration and acceleration of the centre of mass. Due to pure rolling these latter ones are not independent: $a_C = \kappa \cdot R$

Euler's first law in horizontal and vertical directions:

$$20 \cdot 9.81 - N = 20 \cdot 0$$

$$F_f = 20 \cdot a_C$$

The first equation yields $N = 196.2\,\text{N}$ directly but there is no unique solution for the second one.

In order to write also Euler's second law, moments of inertia are needed. About the centre of mass it reads: $I_C = \dfrac{20 \cdot 0.3^2}{2} = 0.9\,\text{kgm}^2$. Pure rolling causes the contact point to be the instantaneous centre of rotation; the moment of inertia about that is $I_0 = \dfrac{3}{2} 20 \cdot 0.3^2 = 2.7\,\text{kgm}^2$. Even though not both of them will be used in the solution, write Euler's second law about for the centre of mass and the contact point as well:

$$\sum M_i^{(C)} : -50 + F_f \cdot 0.3 = 0.9 \cdot (-\kappa)$$

$$\sum M_i^{(0)} : -50 = 2.7 \cdot (-\kappa)$$

From the last one, $\kappa = 18.52\,\text{rad/s}^2 (\curvearrowright)$; thus, $\boxed{a_C = 5.556\,\text{m/s}^2 (\rightarrow)}$

Remark: If the instantaneous centre of rotation is known, it can be regarded as a thumb rule that Euler's second law written about it will result in an equation with relatively few unknowns.

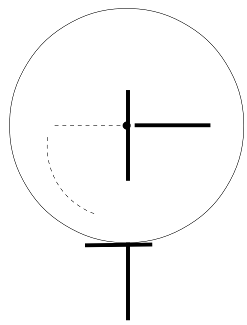
From any of the two remaining equations, $F_f = 111.1\,\text{N} (\rightarrow)$.

From the law of friction, $111.1 < \mu \cdot 196.2 \rightarrow$ $\boxed{\mu > 0.5663}$

Exercise 3

A wheel of mass $m = 20\,\text{kg}$ and radius $R = 0.30$ m is driven by a horizontal force $F = 50\,\text{N}$ of on a horizontal plane.
Find the acceleration of the centre of the wheel in pure rolling.
Find minimum coefficient of friction required for a pure rolling.

Solution



Draw all forces exerted upon the wheel as well as expected acceleration of the center of mass and angular acceleration of the wheel into the figure.

Write two equations of Euler's first law:

$\sum F_{ix}$:

$\sum F_{iy}$:

Calculate moments of inertia about the centre of mass and the instantaneous centre of rotation:

$I_C =$ $\qquad\qquad\qquad\qquad$ $I_0 =$

Write Euler's second law about both points:

$\sum M_i^{(C)}$:

$\sum M_i^{(0)}$:

Solve the system:

$\kappa =$ $\qquad\qquad\qquad\qquad$; thus, $a_C =$

$N =$

$F_f =$

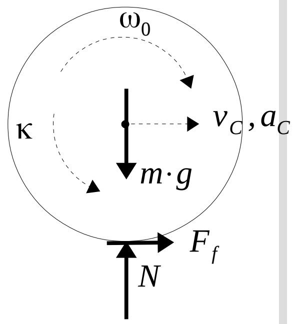
The minimum coefficient of friction required for pure rolling:

$$\mu > \frac{|F_f|}{N} =$$

Example 4

A cylinder of mass $m = 6\,\text{kg}$ and radius $R = 0.2\,\text{m}$ rotates with an angular velocity $\omega_0 = 20\,\text{rad/s}$ when it is put onto a horizontal plane. The velocity of the centre of mass is zero on first touch. What time does it take for the cylinder to reach pure rolling if the coefficient of kinetic friction is $\mu = 0.2$?

Solution



The bottom point of the cylinder is still moving at the instant of first touch, so there is a rolling with slipping. The bottom point moves backwards, causing the force of friction to point forward, but since that is the only force having a moment about the central axis of the cylinder, the sense of both the angular acceleration and horizontal acceleration of the centre of mass is uniquely given (even though these two are now independent).

Euler's first law in vertical direction implies

$6 \cdot 9.81 - N = 6 \cdot 0$, that is, $N = 58.86\,\text{N}$

The force of kinetic friction: $F_f = 0.2 \cdot 58.86 = 11.77\,\text{N}$

Euler's first law in horizontal direction reads

$11.77 = 6 \cdot a_C$, from which the acceleration of the centre of mass is $a_C = 1.962\,\text{m/s}^2 (\rightarrow)$.

Since the bottom point of the cylinder is not an instantaneous centre of rotation, Euler's second law can only be written about the centre of mass:

$11.77 \cdot 0.2 = \frac{6 \cdot 0.2^2}{2} \cdot \kappa$ that yields an angular acceleration, $\kappa = 19.62\,\text{rad/s}^2 (\curvearrowleft)$.

With respect to the preceding results, angular velocity and the velocity of the centre of mass can be written, as far as pure rolling is reached, as follows: $\omega = 20 - 19.62 \cdot t$ and $v_C = 1.962 \cdot t$.

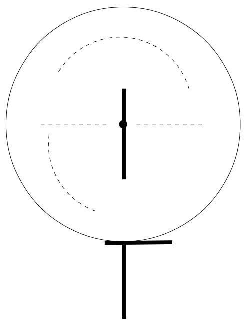
Pure rolling starts when the velocity $(v_C - \omega \cdot R)$ of the bottom of the cylinder becomes zero:

$$1.962 \cdot t - (20 - 19.62 \cdot t)\, 0.2 = 0 \rightarrow \boxed{t = 0.6796\,\text{s}}$$

Exercise 4

A cylinder of mass $m = 6\,\text{kg}$ and radius $R = 0.2\,\text{m}$ is free of rotation when put onto the horizontal ground. The velocity of its centre of mass is $v_{C0} = 3\,\text{m/s}$ at the instant of first touch. What time does it take for the cylinder to reach pure rolling if the coefficient of kinetic friction is $\mu = 0.2$?

Solution



Draw all forces exerted upon the cylinder into the figure and mark the acceleration of the centre of mass as well as the angular acceleration.

Write two equations of Euler's first law:

$\sum F_{ix}$:

$\sum F_{iy}$:

From the solution of the system,

$N =$ $F_f =$

$a_C =$

Calculate moments of inertia about the centre of mass:

$I_C =$

Write Euler's second law about the centre of mass:

$\sum M_i^{(C)}$:

The angular acceleration is obtained from it as

$\kappa =$

The centre of mass of the cylinder performs auniform accelerating motion until pure rolling is reached. Moreover, the simultaneous rotational motion also accelerates uniformly; thus, time functions of velocity of the centre of mass and of angular velocity are as follows:

$v(t) =$ $\omega(t) =$

Pure rolling starts when the velocity $(v_C - \omega \cdot R)$ of the bottom of the cylinder becomes zero;

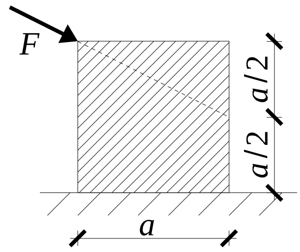
thus, the required time is $t =$

Tip-over stability analysis

Tipping over of bodies on a flat surface is a phenomenon closely related to rolling. Considering the finite domain of contact, the normal force is not concentrated any longer but rather distributed over a length or surface. Since that force can only be compression, its resultant should act inside the domain of contact in any case; in limit it is located at the border of the domain. In a tip-over analysis one must check moments causing tipping against those tending to prevent the body from it (at the same time it is assumed that the point of tipping does not slip; this must be checked after the calculation having been completed).

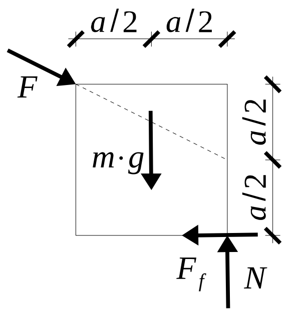
Example 5

Find force F required for the tip over of a body of side length $a=3\,\mathrm{m}$ and mass $m=50\,\mathrm{kg}$ laid on a horizontal surface if the direction of the force is as indicated in the figure.



Solution

The figure illustrates all forces exerted upon the body at the instant of tip over. Since a tip over starts even at infinitesimally small clockwise angular acceleration, the solution will be obtained in limit from an equation (that is why the real rotating motion and therefore the moment of inertia of the body can be left out of consideration).



Euler's second law when tip over starts reads

$$50\cdot 9.81\cdot 1.5-F\cdot\frac{3}{\sqrt{3^2+1.5^2}}\cdot 1.5=I_N\cdot 0 \quad ,$$

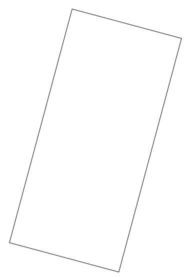
which solves to $\boxed{F=548.4\,\mathrm{N}}$.

Remark:

It was assumed that the corner of contact does not slip, that is, its acceleration is zero in all directions. By writing Euler's first law in two directions, two equations are set up that serve to obtain the force of friction and normal force as well. Minimum coefficient of friction is obtained from the law of friction afterwards.

Exercise 5

A concrete block of height 1 m and width 30 cm weighs $m=180$ kg. The block is laid on a slope of inclination $\alpha=20°$ at its shorter side. Check the block for tipping over.



Solution

There are alternative solutions to the problem. A dynamical approach is taken first: the sense of angular acceleration of the block about the axis of tipping (the bottom right corner) is checked first. If it accelerates upwards, one can conclude that the normal force does not pass through the corner but is offset to the left and the block actually does not move at all. Due to the opposite sense of acceleration; however, the block tips over.

Draw all forces exerted upon the block into the figure with the assumptions taken above.

Write Euler's second law about the assumed centre of rotation (the moment of inertia is yet unknown but is surely positive; the moment of weight should be resolved for convenience into components parallel and perpendicular to the slope):

$$\sum M_i^{(N)}:$$

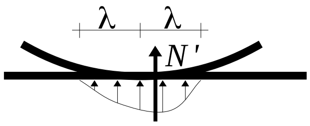
The sign (and thus, the sense) of angular acceleeration is obtained as:

$$0 \quad \kappa \; (\quad)$$

Conclusion: the block ....

Rolling resistance

A rolling wheel, in fact, is not a rigid body. As a result of that, a line rather than a point of contact is formed; accordingly, the normal force is not concentrated to a point, not even it is required to remain centred at the idealized contact as the resultant of a distributed force. When looking for an equilibrium, the force is required only to remain inside the domain of contact. In the case of a rolling motion, an equivalent force-couple system for the normal force is found such that the force component is set back to the ideal point of contact and is completed by a couple called the *moment of rolling resistance*. The sense of that moment is opposite to the angular velocity, its magnitude depends on the normal force and a factor λ of rolling resistance (characteristic of the bodies in contact) as $\Gamma = \lambda \cdot N$.

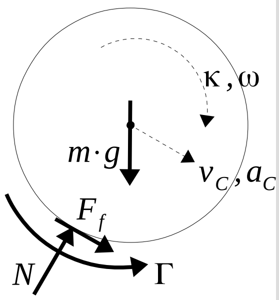


Example 6

A cylinder of radius $r = 0.25$ m is left on its own in rest on the top of a slope with an inclination of $\alpha = 30°$. The cylinder starts moving downwards with a pure rolling motion, its factor of rolling resistance $\lambda = 0.04$ m.
Find the distance needed for the centre of the cylinder to reach the velocity $v_C = 10$ m/s.

Solution

The figure to the right shows all forces exerted upon the cylinder under the assumption of that it rolls without slipping downwards. It means that angular acceleration and angular velocity are both clockwise, while the moment Γ of rolling resistance has an opposite sense with a magnitude $\Gamma = \lambda \cdot N$.



Pure rolling makes the instantaneous centre to be known as the point of application of force *N*, now thought to pass through the centre of the cylinder. For the same reason, $a_C = \kappa \cdot r$, showing the dependence of acceleration and angular acceleration on each other.

Write Euler's first law in directions parallel and perpendicular to the slope:

$$\sum F_{i\swarrow} : m \cdot 9.81 \cdot \cos 30° - N = m \cdot 0$$

$$\sum F_{i\searrow} : m \cdot 9.81 \cdot \sin 30° + F_f = m \cdot a_C$$

From the first one we have $N = 8.496 \cdot m$, so the rolling resistance is $\Gamma = 0.3398 \cdot m$.

Euler's second law can now be written for both possible points:

$$\sum M_i^{(C)} : F_f \cdot 0.25 + 0.3398 \cdot m = \frac{m \cdot 0.25^2}{2} \cdot (-\kappa)$$

$\sum M_i^{(0)}: -m\cdot 9.81\cdot \sin 30°\cdot 0.25 + 0.3398\cdot m = \frac{3m\cdot 0.25^2}{2}\cdot(-\kappa)$

After division by m in the second equation, $\kappa = 9.455\,\text{rad/s}^2$

(The result does not depend on mass m, that is why it has not been given.)

Acceleration of the centre due to pure rolling: $a_C = 9.455\cdot 0.25 = 2.364\,\text{m/s}^2$.

The distance required to reach the desired velocity with the acceleration above:

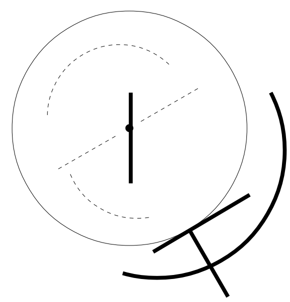
$10^2 = 0^2 + 2\cdot 2.364\cdot s \rightarrow$ $\boxed{s = 21.15\,\text{m}}$

Remark: The force of static friction could be obtained from either of two unused equations (now as a function of m). The minus sign of the result $F_f = -2{,}541\cdot m$ refers to a force opposite the assumed direction. In a folowing step it could also be found that the minimum coefficient of static friction needed for a pure rolling is $\mu = 0.2991$.

Exercise 6

A cylinder of radius $r = 0.25\,\text{m}$ and mass $m = 8\,\text{kg}$ rolls up against the slope of inclination $\alpha = 30°$ without slipping. Initial velocity of the centre of the cylinder is $v_{C1} = 14\,\text{m/s}$. Find the distance covered by the cylinder if the factor of rolling resistance is $\lambda = 0.04\,\text{m}$.

Solution



Draw all forces exerted upon the body, as well as kinematic variables of the centre of mass and expected senses of accelerations into the figure.

Write Euler's first law in directions parallel and perpendicular to the slope:

$\sum F_{i\searrow}:$

$\sum F_{i\swarrow}:$

From the first equation we get

$N =$ $\Gamma =$

Calculate moments of inertia for both the point of contact and instantaneous centre of rotation:

$I_C =$ $I_0 =$

Write Euler's second law for both possible points:

$\sum M_i^{(C)}:$

$\sum M_i^{(0)}:$

Observe that only the angular acceleration is unknown in the second one. Solve that equation:

$\kappa =$

Acceleraation of the centre of mass due to pure rolling:

$a_C =$

The distance until stop calculated from initial and final velocity and acceleration of the centre:

Kinetics of rigid bodies

In a way quite similar to that shown in kinetics of particles, there can also be derived some theorems for motions of rigid bodies that concern initial and final instants of the motion only. The present lesson gives a review on those theorems together with some associated concepts; the discussion is completed by examples as usual.

The impulse-momentum theorem

Euler's first law states that the centroid of a rigid body moves exactly like a particle under the same loads. Analogously to material particles, linear momentum and a theorem of its change can also be interpreted for a rigid body.

Def.: *Linear momentum* of a body equals the product of its mass *m* and the velocity $\boldsymbol{v}_C$ of its centre of mass.

Theorem: The change of linear momentum of a rigid body in a time period equals the impulse of forces exerted on the body in the same time period. Written as a formula, $m\cdot(\boldsymbol{v}_{C2}-\boldsymbol{v}_{C1})=\int_{t_1}^{t_2}\boldsymbol{R}\,\mathrm{d}t$.

It can be seen from the theorem that torques have no influence on the motion of the centre of mass of the body.

The theorem is applicable in cases similar to those discussed in kinetics of patrticles.

The angular impulse-momentum theorem

Def.: *Angular momentum* of a body about an axis equals the product of its moment of inertia about the same axis and its angular velocity.

The *angular impulse-momentum theorem* is stated in two versions (about the central axis and about the axis of rotation) separately:

Theorem: The change of angular momentum of a rigid body about the *central axis* (i.e., axis passing through the centre of mass) in a time period equals the integral of moments of forces exerted on the body about the same axis. Written as a formula, $I_C(\omega_2-\omega_1)=\int_{t_1}^{t_2}M_C\,\mathrm{d}t$.

Theorem: The change of angular momentum of a rigid body about the *fixed axis of rotation* in a time period equals the integral of moments of forces exerted on the body about the same axis. Written as a formula, $I_0(\omega_2-\omega_1)=\int_{t_1}^{t_2}M_0\,\mathrm{d}t$

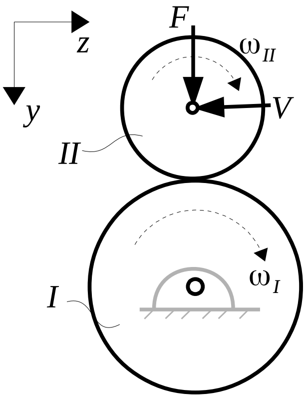
Time integration here still simplifies to a multiplication by time if moments are constant.

This theorem can also be obtained by a simple translation of linear and angular terms of motion according to our dictionary: mass, velocity and resultant force are replaced by the moment of inertia, angular velocity and resultant torque, respectively.

This reference to the dictionary may also help in finding the cases of application of the theorem. Since neither the angle of rotation nor the angular velocity appears in the statement, which implies its use for rotational motions where all but one of the other variables are known.

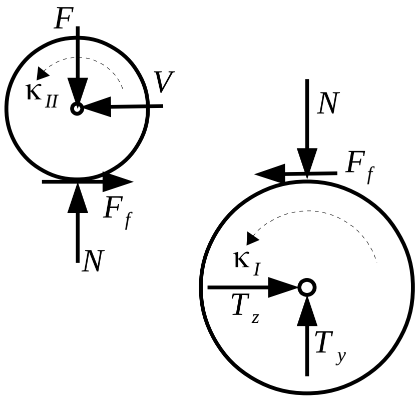
Example 1

A cylinder of mass $M_I = 30\,\text{kg}$ and radius $R_I = 0.6\,\text{m}$ rotates about its fixed vertical axis with an angular velocity $\omega_I = 100\,\text{rad/s}$ (see top view in the figure).
This cylinder is stopped by another one of mass $M_{II} = 60\,\text{kg}$ and radius $R_{II} = 0.45\,\text{m}$ (rotating in the same sense with an angular velocity $\omega_{II} = 200\,\text{rad/s}$) in a way that a force $F = 300\,\text{N}$ is applied at the axis of cylinder II in direction y, as well as another force V in direction z is applied to prevent the same axis from sliding.
The coefficient of kinetic friction between cylinders is $\mu = 0.2$.
Find the time until cylinder I stops.



Solution

Theorems are written according to forces acting on each body; those forces are shown in the figure to the right. Forces in direction *y* are as follows: the given F, the normal force and a component of the force exerted by the axis (T_y). In direction *z*, relative velocities at the point of contact must be observed first: cylinders *I* and *II* move right and left, respectively. Forces of kinetic friction are always opposite; let the force acting on the axes of cylinders *I* and *II* be denoted by T_z and V, respectively.



The top cylinder does not move in direction *y*, so Euler's first law reads now as

$N - 300 = 60 \cdot 0 \rightarrow N = 300\,\text{N}$, hence the force of friction is $F_f = 0.2 \cdot 300 = 60\,\text{N}$.

The bottom cylinder is rotated only by the friction about the axis of rotation. Moment of inertia about that axis is $I_0 = \dfrac{30 \cdot 0.6^2}{2} = 5.4\,\text{kgm}^2$. The angular impulse-momentum theorem for the same axis can now be written between the starting and final instant of braking as

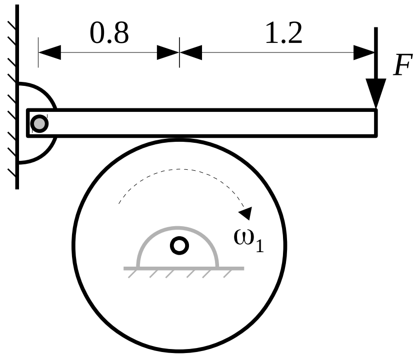
$$\curvearrowright : 5.4 \cdot (0 - 100) = -60 \cdot 0.6 \cdot t\ ;$$

the solution is $\boxed{t = 15\,\text{s}}$.

It can be further used to find angular velocity of the top cylinder still from the same theorem written about its axis of rotation: $\curvearrowright : \dfrac{60 \cdot 0.45^2}{2}(\omega - 200) = -60 \cdot 0.45 \cdot 15 \rightarrow\ \omega = 133.3\,\text{rad/s}(\curvearrowright)$, showing that its original sense of rotation is still preserved. This means that the sense of friction is taken into account correctly, the time calculated above is the solution indeed.

Remark: 'stopping' means here a single instant, cylinder *I* starts then rotating in opposite sense immediately.

Exercise 1

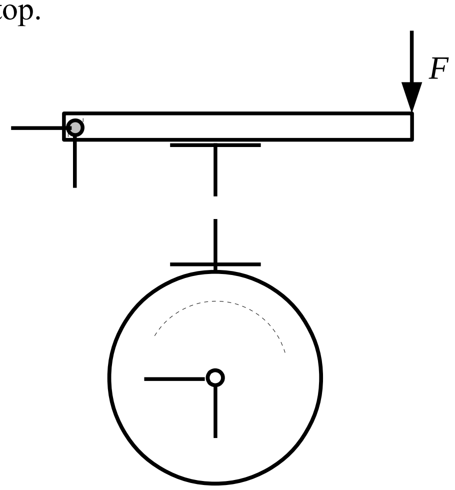


A cylinder of mass $m=30\,\text{kg}$ and radius $R=0.6\,\text{m}$ rotates witha an angular velocity $\omega_1=100\,\text{rad/s}$. The cylinder is stopped by a force $F=300\,\text{N}$ applied at the end of a rod of length $l=2\,\text{m}$ (the rod can rotate about a fixed axis at its other end, the weight of the rod can be neglected). The coefficient of kinetic friction between the rod and the cylinder is $\mu=0.2$. Find the time elapsed until stop.

Solution

The calculation of changes in the motion of the cylinder requires all forces acting on rigid members to be known: the sense of any forces are unchanged in any time instant t until stop.

Draw all forces with their assumed senses into the diagram.



The rod stands still; thus, the angular impulse-momentum theorem can be written about the centre of rotation as an equation with zero on its right hand side. Write this equation assuming that the rod is narrow enough for the rotational effect of friction about its pinpoint be neglected:

$$\sum M_i:$$

This equation can be solved for the normal (tightening) force between the rod and the cylinder:

$$N=$$

The force of kinetic friction: $F_f=$

Since the force of friction is the only one that rotates the cylinder about its axis, the angular impulse-momentum theorem can be written about that axis passing through the centre of mass. The corresponding moment of inertia is a follows:

$$I_C=$$

The theorem should be written for the entire time interval of braking:

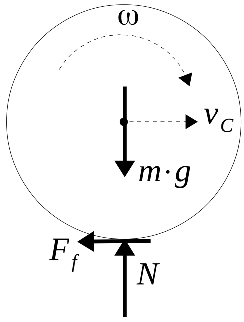
It can be solved for the time until stop:

$$t_2=$$

Example 2

A cylinder of mass $m=6\,\text{kg}$ and radius $R=0.2\,\text{m}$ is free of rotation when put onto the horizontal ground. The velocity of its centre of mass is $v_{C0}=3\,\text{m/s}$ at the instant of first touch. What time does it take for the cylinder to reach pure rolling if the coefficient of kinetic friction is $\mu=0.2$?

Solution



The bottom point of the cylinder moves on touch, which results in a rolling with slipping in the initial instant. If the bottom point of the cylinder moves ahead and is therefore acted upon a force of friction backwards. Since this the only horizontal force that can rotate about the centre of mass, it defines both the sense of horizontal and angular acceleration. (Due to slipping; however, they are now independent.)

From Euler's first law in vertical direction,

$$6 \cdot 9.81 - N = 6 \cdot 0\ ,$$

which yields $N = 58.86\,\mathrm{N}$

and then $F_f = 0.2 \cdot 58.86 = 11.77\,\mathrm{N}$

The impulse-momentum theorem can be written in horizontal direction as follows:

$$6 \cdot (v_{C2} - 3) = -11.77 \cdot t \qquad \rightarrow \qquad v_{C2} = 3 - 1.962 \cdot t$$

The bottom point of the cylinder is NOT the instantaneous centre of rotation but the angular impulse-momentum theorem can still be written about the centre of mass:

$$\curvearrowright : \frac{6 \cdot 0.2^2}{2} \cdot (\omega_2 - 0) = 11.77 \cdot 0.2 \cdot t \qquad \rightarrow \qquad \omega_2 = 19.62 \cdot t$$

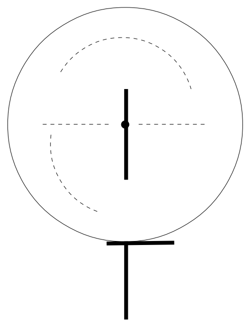
Pure rolling is reached when the velocity $v_{C2} - \omega_2 \cdot R$ of the bottom point of the cylinder reduces to zero, that is, $v_{C2} = \omega_2 \cdot R$:

$$3 - 1.962 \cdot t = 19.62 \cdot t \cdot 0.2 \rightarrow \boxed{t = 0.5097\,\mathrm{s}}$$

Exercise 2

A cylinder of mass $m = 6\,\mathrm{kg}$ and radius $R = 0.2\,\mathrm{m}$ rotates with an angular velocity $\omega_0 = 20\,\mathrm{rad/s}$ when it is put onto a horizontal plane. The velocity of the centre of mass is zero on first touch. What time does it take for the cylinder to reach pure rolling if the coefficient of kinetic friction is $\mu = 0.2$?

Solution



Draw all forces exerted upon the cylinder into the figure and mark the acceleration of the centre of mass as well as the angular acceleration.

Write Euler's first law in a vertical direction:

$$\sum F_{iy} :$$

The solution:

$$N = \qquad\qquad\qquad\qquad F_f =$$

Write now the impulse-momentum theorem in the direction of translation and express velocity as a function of time:

When rolling with slipping, the instantaneous centre of rotation is subject to change, that is why the angular impulse-momentum theorem can only be written about the centre of mass. Calculate the moment of inertia about the axis passing through the centre of mass:

$I_C =$

Write the angular impulse-momentum theorem about the centre of mass and express angular velocity from it:

$\sum M_i^{(C)}$:

When pure rolling starts, $v_{C2} - \omega_2 \cdot R = 0$, that is:

yielding the time elapsed until then as $t_2 =$

The work-energy theorem

Kinetic energy

Kinetic energy of a rigid body can be obtained based on a translational part (given by the velocity of the centre of mass) and a rotational part (about the centre of mass): $T = \frac{1}{2} m \cdot v_C^2 + \frac{1}{2} I_C \cdot \omega^2$.

The parallel axis (or Steiner's) theorem, mentioned in relation with moments of inertia, can be used to prove that kinetic energy of a body in a general plane motion can also be obtained directly from the moment of inertia about its instantaneous centre of rotation: $T = \frac{1}{2} I_0 \cdot \omega^2$. (Because the velocity of the centre of mass is $v_C = \omega \cdot r_{C0}$ where r_{C0} is the distance between the centre of mass and the instantaneous centre of rotation and $m \cdot r_{C0}^2 + I_C = I_0$.)

Mechanical work

The work of forces acting on a rigid body is calculated as in the case of particles. The work of torques can also be interpreted as the sum of works of forces that constitute an equivalent couple. Their works done on the translational part of the motion are the negatives of each other, and the sum of elementary works done on elementary translations can be proved to be equal to the product of the torque and the angle of rotation. Thus, the work of a constant moment *M* is $L_{1-2} = M \cdot (\varphi_2 - \varphi_1)$.

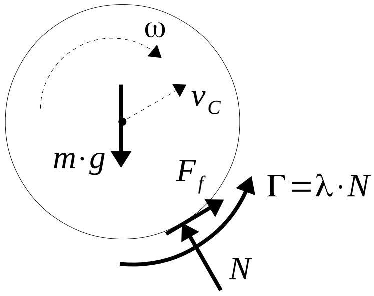
The work-energy theorem

The theorem can still be written in the form $T_2 - T_1 = L_{1-2}$ but the interpretation of terms is slightly modified with respect to that learnt for particles. Kinetic energy components T_i also comprise effects of rotational motion, whereas the term L_{1-2} means the sum of works of (external) forces and torques between the initial and final configuration of the motion.

Example 3

A cylinder of radius $r=0.25$ m and mass $m=8$ kg rolls up against the slope of inclination $\alpha=30°$ without slipping. Initial velocity of the centre of the cylinder is $v_{C1}=14$ m/s.
Find the distance covered by the cylinder if the factor of rolling resistance is $\lambda=0.04$ m.

Solution



The figure to the right shows all forces exerted upon the cylinder in a general instant of motion. Pure rolling implies that $v_C=\omega\cdot r$, and the moment Γ of rolling resistance is counterclockwise.

From a resolution perpendicular to the slope:

$$N-8\cdot 9.81\cdot\cos 30°=0 \rightarrow \quad N=67.97\ \text{N}\ ,$$

which gives the moment of rolling resistance:

$$\Gamma=67.97\cdot 0.04=2.719\ \text{Nm}$$

Kinetic energy is zero when the cylinder stops ($T_2=0$). In the initial configuration, kinetic energy can be obtained in two different ways:

$$T_1=\frac{1}{2}8\cdot 14^2+\frac{1}{2}\frac{8\cdot 0.25^2}{2}\left(\frac{14}{0.25}\right)^2=1176\ \text{J} \qquad \text{(translation plus rotation about the centre of mass)}$$

$$T_1=\frac{1}{2}\frac{3}{2}8\cdot 0.25^2\left(\frac{14}{0.25}\right)^2=1176\ \text{J} \qquad \text{(rotation about the instantaneous centre)}$$

The work of forces exerted on the body as well as of the moment of rolling resistance should be obtained with the following considerations. Forces perpendicular to the slope perform zero work; the bottom point has no velocity, so friction performs no work either. Pure rolling ensures that a distance *s* along the slope means an angle *s*/*R* (rad) of rotation of the cylinder: this is the angle the moment of rolling resistance performs a (negative) work on. The work-energy theorem states that

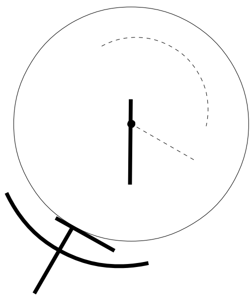
$$0-1176=-8\cdot 9.81\cdot\sin 30°\cdot s-2.719\cdot\frac{s}{0.25}\ ,$$

from which the distance covered until stop is $\boxed{s=23.47\ \text{m}}$

Exercise 3

A cylinder of radius $r=0.25$ m is left on its own in rest on the top of a slope with an inclination of $\alpha=30°$. The cylinder starts moving downwards with a pure rolling motion, its factor of rolling resistance $\lambda=0.04$ m.
Find the distance needed for the centre of the cylinder to reach the velocity $v_C=10$ m/s.

Solution



Draw all forces exerted upon the body, as well as kinematic variables of the centre of mass and expected senses of accelerations into the figure.

Write Euler's first law in the direction perpendicular to the slope:

$$\sum F_{i\swarrow}:$$

From this we get

$N=$ $\Gamma=$

Calculate moments of inertia for both the point of contact and instantaneous centre of rotation:

$I_C=$ $I_0=$

Write the kinetic enerqy both at the beginning and end of the motion:

$T_1=$

$T_2=$

Distances (angles) that individual forces (moments) do work on:

$m\cdot g\cdot\cos\alpha:$ $m\cdot g\cdot\sin\alpha:$

$N:$ $\Gamma:$

$F_f:$

The work-energy theorem:

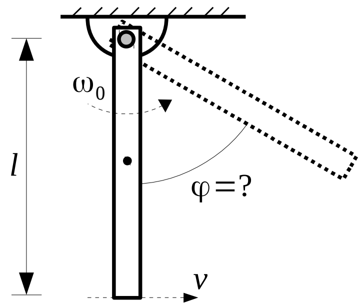
$T_2-T_1=L_{1-2}:$

The distance that was looked for is then

$s=$

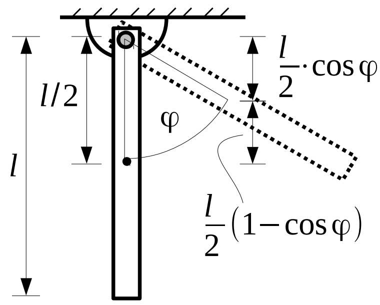
Example 4

A rod of mass $m=5\,\text{kg}$ and length $l=1.6\,\text{m}$ swings about a pinned support. While passing the bottom vertical configuration, the velocity of its lowest point is $v=3\,\text{m/s}$. Find the maximum angle φ of deviation.



Solution

Angular acceleration could be found from forces exerted on the rod for any particular angle of deviation but κ would not be constant and the solution of the resultant differential equation would not be easy at the present level of discussion, that is why the work-energy theorem is used here.



The kinetic energy at the instant of stop is $T_2=0$ J, while for the bottom vertical configuration it can be directly obtained from the rotation about the pin. At the same time, angular velocity is calculated from the velocity of the bottom point:

$$\omega_0=v/l=3/1.6=1.875\,\text{rad/s}$$

Kinetic energy is therefore $T_1=\frac{1}{2}\frac{5\cdot1.6^2}{3}1.875^2=7.5\,\text{J}$

Forces transmitted at the pin do zero work because the pin is fixed. The work of gravity depends on the vertical translation of the centre of mass of the rod and can be found based on the second figure: the translation happens upwards, making the work of a force directed downwards to be negative. The theorem now reads:

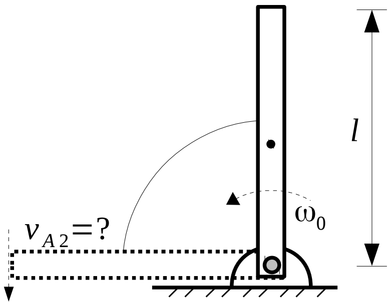
$$0-7.5=-5\cdot 9.81\cdot\frac{1.6}{2}(1-\cos\varphi),$$

which yields the maximum angular deviation as $\boxed{\varphi=36.01°}$

Exercise 4

A rod of mass $m=8\,\mathrm{kg}$ and length $l=1.8\,\mathrm{m}$ is supported by a pin at its bottom. The rod passes its upright position with an angular velocity $\omega_0=4$ rad/s.

Find the velocity of the moving endpoint when the rod reaches a horizontal position.



Solution

If the angular velocity of the body in its horizontal position is known, then the velocity at any other point of the body can be obtained.

The body rotates about a fixed axis, so its kinetic energy can be calculated from the formula related to the instantaneous centre of rotation; the corresponding moment of inertia is

$$I_0=$$

Thus, kinetic energy in at the initial instant:

$$T_1=$$

The kinetic energy just before the collision, assuming ω to be the current angular velocity:

$$T_2=$$

The work done by forces on the body:

$$L_{1-2}=$$

The work-energy theorem is applied as

$T_2-T_1=L_{1-2}\rightarrow$

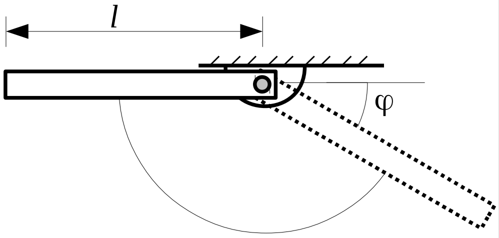
The angular velocity is obtained as $\omega=$

The velocity of the free end of the rod: $v_{A2}=$

Example 5

A rod of mass $m=3\,\mathrm{kg}$ and length $l=0.6\,\mathrm{m}$ supported by a pin at one end is lift to a horizontal position then released.

Find the velocity of the centre of mass as well as the angular velocity when the rod reaches the position at $\varphi=20°$ as shown in the figure



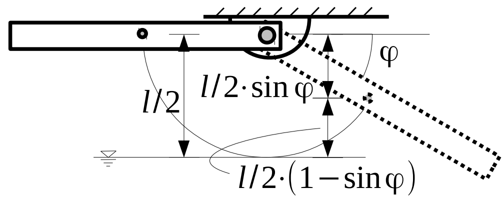
Solution

Since the rod starts from rest, its initial kinetic energy is zero $(T_1=0\,\mathrm{J})$.

Its angular velocity in the final position is ω_2, so kinetic energy is

$$T_2=\frac{1}{2}\frac{3\cdot 0.6^2}{3}\omega_2^2=\frac{1}{2}0.36\cdot\omega_2^2=0.18\,\omega_2^2$$

There is only a displacement under the point of application of weight, so it is the only force performing work in the problem. Calculate that work from potential energy of the force with a base level set to the bottom position of the centre; its initial value is



$$U_1=3\cdot 9.81\cdot 0.3=8.829\,\mathrm{J}$$

In the final configuration,

$$U_2=3\cdot 9.81\cdot(0.3-0.3\cdot\sin 20°)=5.809\,\mathrm{J}\ .$$

The work done by gravity reads: $L_{1-2}=U_1-U_2=8.829-5.809=3.02\,\mathrm{J}$

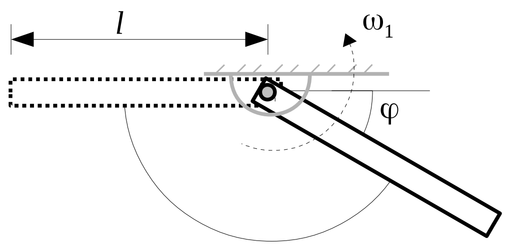
The work-energy theorem implies that: $0.18\,\omega_2^2-0=3.02$,

yielding the angular velocity: $\boxed{\omega_2=4.096\,\mathrm{rad/s}}$

and velocity of the centre of mass: $v_{C2}=4.096\cdot 0.3$ ➔ $\boxed{v_{C2}=1.229\,\mathrm{m/s}}$

Exercise 5

A rod of mass $m=5\,\mathrm{kg}$ and length $l=0.8\,\mathrm{m}$ supported by a pin at one end is released with an angular velocity ω_1 from a position at $\varphi=20°$ as shown in the figure.
Find ω_1 such that the rod could still pass its top vertical position.



Solution

An angular velocity is looked for when the rod just reaches its vertical position. Its kinetic energy at that instant is then

$$T_2=$$

A kinetic energy cannot be smaller as it would be associated with an imaginary velocity.

In the initial configuration; however, the body moves: kinetic energy can be written here as a function of angular velocity. The moment of inertia about the axis of rotation,

$$I_0=$$

The kinetic energy in the initial configuration in terms of ω_1 is written as:

$$T_1=$$

Find vertical component of the translation of the centre of mass:

$$s_y=$$

The work-energy theorem:

that solves to

$\omega_1 =$

Summary

For the sake of clarity, terms, definitions and theorems related to the motion of a particle or rigid body are collected in a table below:

	Particle	Rigid body
kinematic variables:	position: $x(t), \boldsymbol{r}(t)$ velocity: $v(t), \boldsymbol{v}(t)$ acceleration: $a(t), \boldsymbol{a}(t)$	position of the centre of mass: $x_C(t), \boldsymbol{r}_C(t)$ angle of rotation of the body: $\varphi(t)$ velocity of the centre of mass: $v_C(t), \boldsymbol{v}_C(t)$ angular velocity of the body: $\omega(t)$ acceleration of the centre of mass: $a_C(t), \boldsymbol{a}_C(t)$ angular acceleration of the body: $\kappa(t)$
law of motion:	$\sum \boldsymbol{F} = m \cdot \boldsymbol{a}$	$\sum \boldsymbol{F} = m \cdot \boldsymbol{a}_C$ - for the centre of mass $\sum M_C = I_C \cdot \kappa$ - about the centre of mass $\sum M_0 = I_0 \cdot \kappa$ - about the inst. centre of rotation
terms:	linear momentum: $m\boldsymbol{v}$ kinetic energy: $T = \frac{1}{2} m v^2$ work: $L_{1-2} = R(s_2 - s_1)$ potential of gravity: $U_g = mgh$	linear momentum: $m\boldsymbol{v}_C$ - for the centre of mass langular momentum: $I_C \omega$, $I_0 \omega$ $T = \frac{1}{2} m v_C^2 + \frac{1}{2} I_C \omega^2$ - translational + rotational $T = \frac{1}{2} I_0 \omega^2$ - about the inst. centre of rotation $L_{1-2} = R(s_2 - s_1) + M(\varphi_2 - \varphi_1)$ $U_g = mgh_C$
theorems:	I-M: $m(\boldsymbol{v}_2 - \boldsymbol{v}_1) = \boldsymbol{R}(t_2 - t_1)$ W-E: $T_2 - T_1 = L_{1-2}$	I-M: $m(\boldsymbol{v}_{C2} - \boldsymbol{v}_{C1}) = \boldsymbol{R}(t_2 - t_1)$ - centre of mass AI-M: $I_C(\omega_2 - \omega_1) = M_C(t_2 - t_1)$ - c. of mass $I_0(\omega_2 - \omega_1) = M_0(t_2 - t_1)$ - inst. c. of rot. W-E: $T_2 - T_1 = L_{1-2}$

Vector product

Vector product (or cross product, $\boldsymbol{a}\times\boldsymbol{b}$, named after the cross sign of multiplication) of two vectors is a third vector perpendicular to the plane spanned by the first two vectors, its magnitude equals the area of parallelogram spanned by the same two vectors, and is directed according to the handedness of the coordinate system. Because of that direction, the order of vectors in a cross product matters, since $\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$. The above definition implies that unit vectors along axes x, y, z satisfy the following relationships: $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}$, $\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$ and $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=0$. Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be expressed in terms of unit vectors as follows: $\boldsymbol{a}=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k}$ and $\boldsymbol{b}=b_x\boldsymbol{i}+b_y\boldsymbol{j}+b_z\boldsymbol{k}$. Plugging both sums into the cross product then multiplying by terms and using identities on unit vectors we get:

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix}\times\begin{bmatrix}b_x\\b_y\\b_z\end{bmatrix}=\begin{bmatrix}a_y b_z-a_z b_y\\a_z b_x-a_x b_z\\a_x b_y-a_y b_x\end{bmatrix}.$$

As an alternative of memorizing this formula, one can equivalently depart from the expansion of a determinant as follows:

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}.$$

For instance, when expanding a 3-by-3 determinant, each entry of the top row is multiplied by a difference of products of two successive entries available through steps right and down and of two successive entries available through steps left and down. The matrix is interpreted here as a cyclic one: when a left or right border is reached, one can continue at the opposite border. The above figure illustrates this procedure for the top left entry $\boldsymbol{i}$, that is, for the first coordinate of the product vector (green circles refer to positive, red ones to negative products).

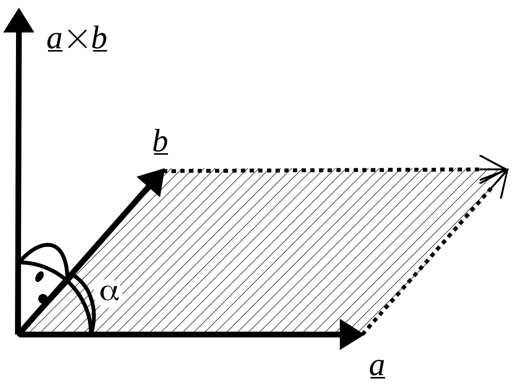
Example 1
Calculate vectors $\underline{a}\times\underline{b}$, $\underline{b}\times\underline{a}$

if $\underline{a}=\begin{bmatrix}12.5\\-5,4\\8.73\end{bmatrix}$, $\underline{b}=\begin{bmatrix}3.9\\-11,3\\6.6\end{bmatrix}$.

Solution

$$\underline{a}\times\underline{b}=\begin{bmatrix}-5.4\cdot6.6-8.73\cdot(-11.3)\\8.73\cdot3.9-12.5\cdot6.6\\12.5\cdot(-11.3)-(-5.4)\cdot3.9\end{bmatrix}=\begin{bmatrix}63.01\\-48.45\\-120.2\end{bmatrix}$$

$$\underline{b}\times\underline{a}=\begin{bmatrix}6.6\cdot(-5.4)-(-11.3)\cdot8.73\\3.9\cdot8.73-6.6\cdot12.5\\(-11.3)\cdot12.5-3.9\cdot(-5.4)\end{bmatrix}=\begin{bmatrix}-63.01\\48.45\\120.2\end{bmatrix}$$



Exercise 1

Calculate vectors $\underline{a}\times\underline{b}$, $\underline{b}\times\underline{a}$ if $\underline{a}=\begin{bmatrix}21.07\\-1.87\\0.0\end{bmatrix}$, $\underline{b}=\begin{bmatrix}42.2\\13.8\\0.0\end{bmatrix}$.

Solution

$$\underline{a}\times\underline{b}=\begin{bmatrix} \\ \\ \end{bmatrix}=\begin{bmatrix} \\ \\ \end{bmatrix}$$

Remark: Since the z component of both vectors is zero, the parallelogran spanned by $\underline{a}$ and $\underline{b}$ lies in the plane xy. The angle between vectors $\underline{a}$ and $\underline{b}$ is $\alpha=\arccos\left(|\underline{a}\cdot\underline{b}|/(|\underline{a}||\underline{b}|)\right)=23.18\,°$, hence the area of parallelogram is $|\underline{a}||\underline{b}|\sin\alpha=369.7$, just equals the length of the product vector $\underline{a}\times\underline{b}$.

$$\underline{b}\times\underline{a}=\begin{bmatrix} \\ \\ \end{bmatrix}=\begin{bmatrix} \\ \\ \end{bmatrix}$$

We emphasize that all discussed vector operations were defined for two or more vectors, followed by computational methods that can be implemented in a given coordinate system. It implies that any change in the coordinate system may influence the numeric solution of the problem; however, its physical interpretation remains the same.

Problems on forces in 3D

Almost all problems discussed so far within this subject were either original planar problems or more complex ones simplified to 2D. Unfortunately, not all 3D problems can be solved using two-dimensional model and formalism; however, a general three-dimensional description is applicable in solutions to common 2D problems as well. The introduction to theory of forces and structures in the space is developed in a way analogous to which was followed in planar force systems: basic operations on forces are revised, possible types of constraints of structures and solution methods to some typical structures are presented. Finally, the procedure of finding internal forces in cross section is also extended to 3D.

Moments and forces in the three-dimensional space

If a force (acting at a given point) is specified in 3D, three signed components are needed in contrast to two components that are sufficient in 2D. (If a force is intended to be given by magnitude and direction, this latter one can be specified by three angles made by the force vector and each coordinate axis; however, the squares of cosines of those angles must add up to 1; thus, there are still three independent scalars only.)

For the specification of a torque in the space, recall that a torque appearing in planar problems could always be interpreted as a moment rotating about an axis perpendicular to the plane. For that reason, a 3D approach requires moments about three axes to be given, so moments have to be dealt with as vectors henceforth (as has already been done in Eample 2 and Exercise 2 of Lecture 4). Viewed down in front of its vector (that is, if the tip of its vector points towards the observer), a torque represents a positive (counterclockwise) rotation.

Resultant of a 3D force system

As has already been noticed in planar problems, two successive steps are necessary for the determination of a unique effect equivalent to a given force system. In the first step, an *equivalent force-couple system* is found that replaces the system by a force passing through a fixed point and a couple associated to it. Having both the force and the couple determined, it is already possible to decide upon the type of resultant and calculate its further properties when necessary.

Finding the equivalent force-couple system at a point (also called *reduction of the system to a point* for brevity) is technically done by the use of resolution equations: force components obtained in this way are independent of the point chosen. The associated moment is obtained from moment equations written about axes passing through the point chosen. These equations should contain signed components of moment vectors that are parallel to the respective axis. With an analogy to 2D calculations, moment of a force about an axis t can also be interpreted as the moment of its component lying within a plane perpendicular to t about the same axis t.

In the decision on the type of resultant, the following rules apply. If the force si zero (i. e., all its components are zero), two cases are possible: either the associated moment is also zero and the force system is in *equilibrium* or the moment is nonzero and that *torque* itself is the resultant of the system. (Keep in mind that moment vectors, unlike forces, are free vectors and can therefore be translated in any sense without changing its physical effect.) If the force (any of its three components) is nonzero, the type of resultant still depends on whether or not the vectors of force and torque can be replaced by an equivalent force as has been done in two dimensions. If the two vectors are perpendicular to each other, then such a replacement is possible and the resultant is a *force*. If, however, the associated moment has a nonzero component parallel to the force vector, replacement by an equivalent force will only eliminate the component of torque perpendicular to the force but the parallel component remains. It means that the resultant of the force system consists of a force and a torque of parallel directions. This resultant is called *wrench* (the quotient of moment and force is often referred to as the pitch of the wrench).

In the above procedure it is crucial to check two vectors for orthogonality. Far the most efficient method is the evaluation of their dot product: a nonzero result implies both that the vectors are not perpendicular and none of them is a zero vector. Based on this property, the following table might help in the process of deciding on the type of resultant.

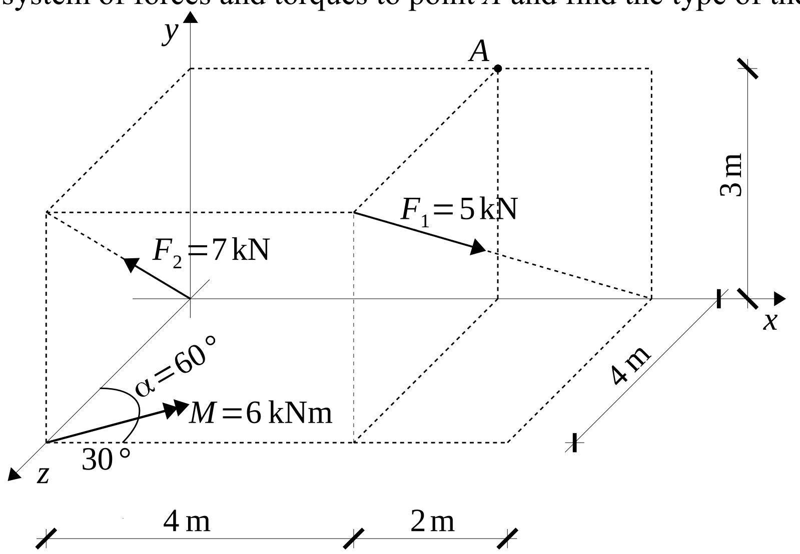
Type of resultant	Properties of the force and the associated moment	Notes
equilibrium	$\boldsymbol{F}=\boldsymbol{0}$, $\boldsymbol{M}=\boldsymbol{0}$	Both vectors must be zero vectors.
torque	$\boldsymbol{F}=\boldsymbol{0}$, $\boldsymbol{M}\neq\boldsymbol{0}$	
force	$\boldsymbol{F}\neq\boldsymbol{0}$, $\boldsymbol{F}\cdot\boldsymbol{M}=0$	Dot product is also zero if $\boldsymbol{M}$ is a zero vector.
wrench	($\boldsymbol{F}\neq\boldsymbol{0}$), $\boldsymbol{F}\cdot\boldsymbol{M}\neq 0$	Dot product would give zero if any of $\boldsymbol{F}$ or $\boldsymbol{M}$ were zero vectors.

If the resultant is either a force or a wrench, yet another step is left: a point on the line of action of the resultant needs to be determined with respect to the point where the system was reduced to. In order to do so, let the associated moment be resolved into components parallel $(\boldsymbol{M}_{\parallel})$ and perpendicular $(\boldsymbol{M}_{\perp})$ to the force. The shift of the line of action measures $k=|\boldsymbol{M}_{\perp}|/|\boldsymbol{F}|$, and its direction should be perpendicular to both the force and the perpendicular component $\boldsymbol{M}_{\perp}$.

Unit vector of that direction of shift can be obtained with the help of the cross product of two vectors: $\dfrac{\boldsymbol{F}\times\boldsymbol{M}_{\perp}}{|\boldsymbol{F}\times\boldsymbol{M}_{\perp}|}$. Because of their orthogonality, the dot product of two vectors in the denominator can be written as an algebraic product of lengths, which yields the vector of shift between lines of action as: $\Delta r=\dfrac{\boldsymbol{F}\times\boldsymbol{M}_{\perp}}{|\boldsymbol{F}||\boldsymbol{M}_{\perp}|}\cdot\dfrac{|\boldsymbol{M}_{\perp}|}{|\boldsymbol{F}|}=\dfrac{\boldsymbol{F}\times\boldsymbol{M}_{\perp}}{|\boldsymbol{F}||\boldsymbol{F}|}$. In addition to that, $\boldsymbol{F}\times\boldsymbol{M}=\boldsymbol{F}\times(\boldsymbol{M}_{\parallel}+\boldsymbol{M}_{\perp})=\boldsymbol{F}\times\boldsymbol{M}_{\parallel}+\boldsymbol{F}\times\boldsymbol{M}_{\perp}=\boldsymbol{0}+\boldsymbol{F}\times\boldsymbol{M}_{\perp}=\boldsymbol{F}\times\boldsymbol{M}_{\perp}$, and the length of a vector equals the square root of dot product taken with itself; the relative position of the line of action of the resultant is given by the vector $\Delta r=\dfrac{\boldsymbol{F}\times\boldsymbol{M}}{\boldsymbol{F}\cdot\boldsymbol{F}}$.

Example 2

Reduce the given system of forces and torques to point A and find the type of the resultant.



Solution

The problem of finding equivalent force-couple system at point A can be formulated as an equivalence statement $(F_A, M_A) \doteq (F_1, F_2, M)$. Components of force F_A are found by writing and solving resolution equations. Components of oblique forces are calculated from the actual length and its projections (onto each coordinate axis) of any segment along the line of action of the force (fractions in the resolution equations below correspond to cosines of angles made by the line of action with each axis). Based on the figure above, the length of oblique segments can be chosen as follows:

$l_1=\sqrt{2^2+3^2+4^2}=5.385\,\mathrm{m}$, $l_2=\sqrt{0^2+3^2+4^2}=5\,\mathrm{m}$.

The resolution equations read:

$$\sum F_{ix}: F_{Ax}=+5\cdot\frac{2}{5.385} \qquad \rightarrow F_{Ax}=+1.857\,\mathrm{kN}$$

$$\sum F_{iy}: F_{Ay}=-5\cdot\frac{3}{5.385}+7\cdot\frac{3}{5} \qquad \rightarrow F_{Ay}=+1.414\,\mathrm{kN}$$

$$\sum F_{iz}: F_{Az}=-5\cdot\frac{4}{5.385}+7\cdot\frac{4}{5} \qquad \rightarrow F_{Az}=+1.886\,\mathrm{kN}$$

Components of the associated moment can be foundfor convenience from moment equations written about each axis passing through point A (for clarity, all forces are now resolved in their point of application shown in the figure):

$\sum M_{ix}^{A} : M_{Ax} = +5 \cdot \frac{3}{5.385} \cdot 4 - 7 \cdot \frac{4}{5} \cdot 3 + 6 \cdot \sin 60° \quad \rightarrow M_{Ax} = -0.4618\,\text{kNm}$

$\sum M_{iy}^{A} : M_{Ay} = +5 \cdot \frac{2}{5.385} \cdot 4 + 7 \cdot \frac{4}{5} \cdot 4 + 0 \quad \rightarrow M_{Ay} = 29.83\,\text{kNm}$

$\sum M_{iz}^{A} : M_{Az} = +0 - 7 \cdot \frac{3}{5} \cdot 4 - 6 \cdot \cos 60° \quad \rightarrow M_{Az} = -19.8\,\text{kNm}$

The equivalent force-couple system is therefore: $\boldsymbol{F}_A = \begin{bmatrix} 1.857 \\ 1.414 \\ 1.886 \end{bmatrix} \text{kN}\,,\ \boldsymbol{M}_A = \begin{bmatrix} -0.4618 \\ 29.83 \\ -19.8 \end{bmatrix} \text{kNm}$

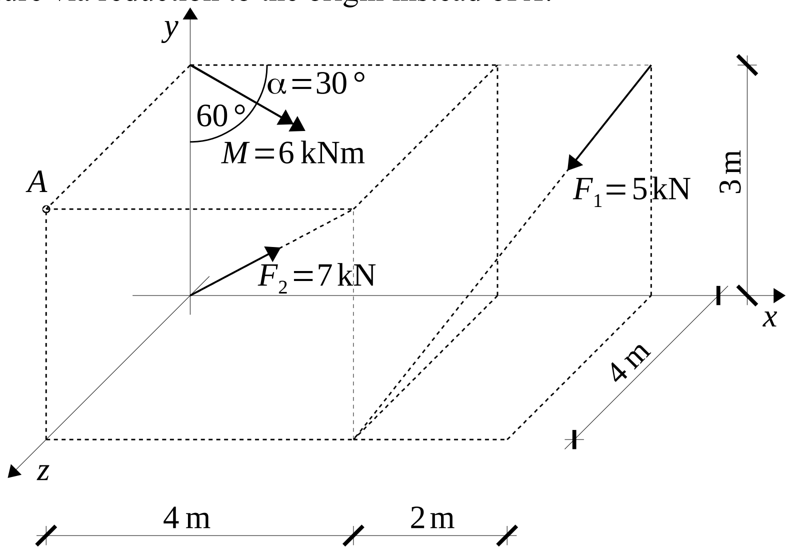
Since the force is not a zero vector, calculate the dot product of the two vectors:

$$\boldsymbol{F}_A \cdot \boldsymbol{M}_A = 1.857 \cdot (-0.4618) + 1.414 \cdot 29.83 + 1.886 \cdot (-19.8) = 3.979\,\text{kN} \cdot \text{kNm}$$

Since this is nonzero, the resultant is a wrench.

Exercise 2

Reduce the given system of forces and torque to point *A* and find the type of the resultant.
Repeat the procedure via reduction to the origin instead of *A*.



Solution 1: Reduction to point A

Given segments of both lines of action are needed in order to find components of oblique forces:

$l_1 =$

$l_2 =$

Equivalence statement: $\doteq$

that makes possible to calculate all three components of the force from resolution equations:

$\sum F_{i..}:$

$\sum F_{i..}:$

$\sum F_{i..}:$

The associated moment is obtained from three moment equations:

$$\sum M_{i..}:$$
$$\sum M_{i..}:$$
$$\sum M_{i..}:$$

The two calculated vectors written in components:

$$\boldsymbol{F}_A = \left[\quad \right], \ \boldsymbol{M}_A = \left[\quad \right]$$

Their dot product for deciding on the type of resultant:

$$\boldsymbol{F}_A \cdot \boldsymbol{M}_A =$$

Thus, the resultant is:

Solution 2: Reduction to the origin

Three components of the force still from resolution equations are as follows:

$$\sum F_{i..}:$$
$$\sum F_{i..}:$$
$$\sum F_{i..}:$$

Three moment equations used for obtaining each component of the associated moment read:

$$\sum M_{i..}:$$
$$\sum M_{i..}:$$
$$\sum M_{i..}:$$

The calculated vectors are

$$\boldsymbol{F}_0 = \left[\quad \right], \ \boldsymbol{M}_0 = \left[\quad \right]$$

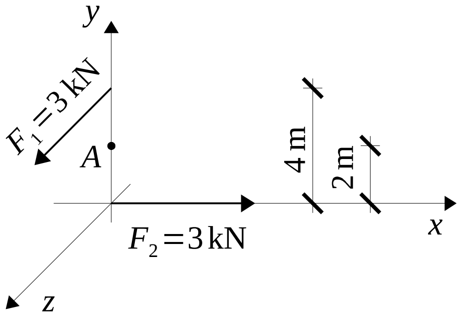
Their dot product for deciding on the type of resultant:

$$\boldsymbol{F}_0 \cdot \boldsymbol{M}_0 =$$

The dot product should be equal to that obtained in the previous solution; thus, the same conclusion should be drawn. The resultant is:

Example 3

Identify the type of resultant of two forces shown in the figure.



Solution 1

If the equaivalent force-couple system is found at the origin, then $(F_0, M_0) \doteq (F_1, F_2)$ implies resolution equations that solve to $\sum F_{ix}: F_{0x}=+3\,\text{kN}$, $\sum F_{iy}: F_{0y}=0\,\text{kN}$, $\sum F_{iz}: F_{0z}=+3\,\text{kN}$.

Moment equations are written about the coordinate axes:

$$\sum M_{ix}: M_{0x}=0+3\cdot 4 \quad \rightarrow M_{0x}=+12\,\text{kNm}$$
$$\sum M_{iy}: M_{0y}=0$$
$$\sum M_{iz}: M_{0z}=0$$

The two vectors written out by components: $\boldsymbol{F}_0=\begin{bmatrix}3\\0\\3\end{bmatrix}\text{kN}, \boldsymbol{M}_0=\begin{bmatrix}12\\0\\0\end{bmatrix}\text{kNm}$,

their dot product is $\boldsymbol{F}_0 \cdot \boldsymbol{M}_0=3\cdot 12+0\cdot 0+3\cdot 0=36\neq 0$ showing that the resultant is a wrench.

Note: If an equivalent force-couple system was found first at any endpoint, say *A*, of the *normal transversal* of two lines of action (i.e., the shortest segment, say *AB*, connecting two skew lines), then the associated moment will be parallel to the plane spanned by vectors parallel to the forces but perpendicular to the force at point *B*. Since the sum of two forces will not be paralel to any original force, so will not be perpendicular to the associated moment either: their dot product cannot give zero. It can be therefore formulated as a general rule that the resultant of two forces in skew lines of action is always a wrench.

Solution 2

If the equivalent force-couple system is found at point *A*, then $(F_A, M_A) \doteq (F_1, F_2)$ implies resolution equations of the same content as those of Solution 1.

Moment equations are now written about axes passing through point *A* in coordinate directions:

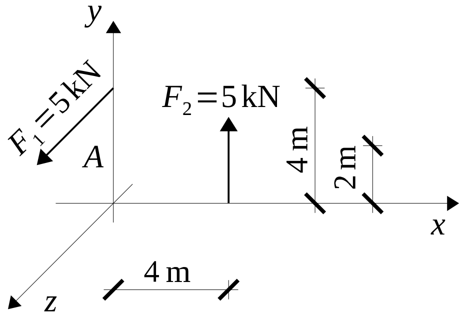
$$\sum M_{ix}^A: M_{Ax}=0+3\cdot 2 \quad \rightarrow M_{Ax}=+6\,\text{kNm}$$
$$\sum M_{iy}^A: M_{Ay}=0$$
$$\sum M_{iz}^A: M_{Az}=3\cdot 2+0 \quad \rightarrow M_{Az}=+6\,\text{kNm}$$

The two vectors are: $\boldsymbol{F}_A=\begin{bmatrix}3\\0\\3\end{bmatrix}\text{kN}, \boldsymbol{M}_A=\begin{bmatrix}6\\0\\6\end{bmatrix}\text{kNm}$.

It is quite straightforward to see now that these vectors are parallel, that is, their resultant is a wrench where the line of action of the force in that wrench passes through point *A*. If the dot product is evaluated nevertheless $(\boldsymbol{F}_A \cdot \boldsymbol{M}_A=3\cdot 6+0\cdot 0+3\cdot 6=36\neq 0)$, it will not only be found nonzero but one can also see that it equals the former product $\boldsymbol{F}_0 \cdot \boldsymbol{M}_0$. It is true in general: the dot product of force and associated moment in a force-couple system equivalent to a given system of forces is constant, irrespective of the point the reduction has been done to.

Exercise 3

Determine the type of resultant of two forces shown.



Solution

Find the point of reduction and write the corresponding equivalence statement:

Calculate force components from resolution equations:

$\sum F_{ix}$:

$\sum F_{iy}$:

$\sum F_{iz}$:

Calculate components of the associated moment:

$\sum M_{ix}$:

$\sum M_{iy}$:

$\sum M_{iz}$:

Write both vectors by components and decide on the type of resultant:

$$\boldsymbol{F} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

The resultant is: